CHAPTER

4

The Time Value of Money

NOTATION

r	interest rate
C	cash flow
FV_n	future value on date n
PV	present value; annuity spreadsheet notation for the initial amount
C_n	cash flow at date n
N	date of the last cash flow in a stream of cash flows
NPV	net present value
P	initial principal or deposit, or equivalent present value
FV	future value; annuity spreadsheet notation for the extra final payment
g	growth rate
$NPER$	annuity spreadsheet notation for the number of periods or dates of the last cash flow
$RATE$	annuity spreadsheet notation for interest rate
PMT	annuity spreadsheet notation for cash flow
IRR	internal rate of return
PV_n	present value on date n

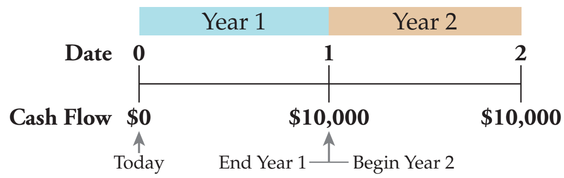
As discussed in Chapter 3, to evaluate a project a financial manager must compare its costs and benefits. In most cases, the cash flows in financial investments involve more than one future period. For example, early in 2003, the Boeing Company announced that it was developing the 7E7, now known as the 787 Dreamliner, a highly efficient, long-range airplane able to seat 200 to 250 passengers. Boeing's project involves revenues and expenses that will occur many years or even decades into the future. The first commercial flight of a 787 was not until October, 2011, flown by All Nippon Airways (ANA); Air Canada's first 787 is scheduled to fly in 2014. Components of the 787 Dreamliner planes will be produced in Boeing plants in the United States and around the world (including Boeing Winnipeg in Canada). How can financial managers evaluate a project such as the 787 Dreamliner airplane?

As we learned in Chapter 3, Boeing should make the investment in the 787 Dreamliner if the *NPV* is positive. Calculating the *NPV* requires tools to evaluate cash flows lasting several periods. We develop these tools in this chapter. The first tool is a visual method for representing a stream of cash flows: the *timeline.* After constructing a timeline, we establish three important rules for moving cash flows to different points in time. Using these rules, we show how to compute the present and future values of the costs and benefits of a general stream of cash flows, and how to compute the *NPV*. Although these techniques can be used to value any type of asset, certain types of assets have cash flows that follow a regular pattern. We develop shortcuts for *annuities*, *perpetuities*, and other special cases of assets with cash flows that follow regular patterns.

4.1 THE TIMELINE

We begin our look at valuing cash flows lasting several periods with some basic vocabulary and tools. We refer to a series of cash flows lasting several periods as a **stream of cash flows**. We can represent a stream of cash flows on a **timeline**, a linear representation of the timing of the expected cash flows. Timelines are an important first step in organizing and then solving a financial problem. We use them throughout this text.

To illustrate how to construct a timeline, assume that a friend owes you money. He has agreed to repay the loan by making two payments of $10,000 at the end of each of the next two years. We represent this information on a timeline as follows:

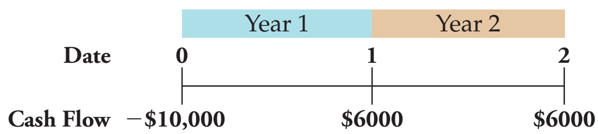


Date 0 represents the present. Date 1 is one year later and represents the end of the first year. The $10,000 cash flow below date 1 is the payment you will receive at the end of the first year. Date 2 is two years from now; it represents the end of the second year. The $10,000 cash flow below date 2 is the payment you will receive at the end of the second year.

You will find the timeline most useful in tracking cash flows if you interpret each point on the timeline as a specific date. The space between date 0 and date 1 then represents the time period between these dates, in this case, the first year of the loan. Date 0 is the beginning of the first year, and date 1 is the end of the first year. Similarly, date 1 is the beginning of the second year, and date 2 is the end of the second year. By denoting time in this way, date 1 signifies *both* the end of year 1 and the beginning of year 2, which makes sense since those dates are effectively the same point in time.[1]

In this example, both cash flows are inflows. In many cases, however, a financial decision will involve both inflows and outflows. To differentiate between the two types of cash flows, we assign a different sign to each: Inflows are positive cash flows, whereas outflows are negative cash flows.

To illustrate, suppose you're still feeling generous and have agreed to lend your brother $10,000 today. Your brother has agreed to repay this loan in two instalments of $6000 at the end of each of the next two years. The timeline is:



Notice that the first cash flow at date 0 (today) is represented as −$10,000 because it is an outflow. The subsequent cash flows of $6000 are positive because they are inflows.

So far, we have used timelines to show the cash flows that occur at the end of each year. Actually, timelines can represent cash flows that take place at the end of any time period.

1. That is, there is no real time difference between a cash flow paid at 11:59 P.M. on December 31 and one paid at 12:01 A.M. on January 1, although there may be some other differences such as taxation that we overlook for now.

For example, if you pay rent each month, you could use a timeline like the one in our first example to represent two rental payments, but you would replace the "year" label with "month."

Many of the timelines included in this chapter are very simple. Consequently, you may feel that it is not worth the time or trouble to construct them. As you progress to more difficult problems, however, you will find that timelines identify events in a transaction or investment that are easy to overlook. If you fail to recognize these cash flows, you will make flawed financial decisions. Therefore, we recommend that you approach *every* problem by drawing the timeline as we do in this chapter.

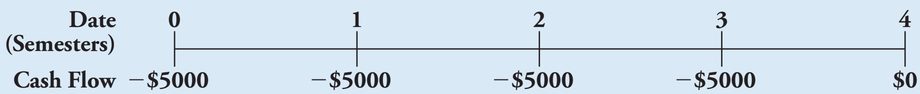
EXAMPLE 4.1 **CONSTRUCTING A TIMELINE**

Problem

Suppose you must pay tuition and residence fees of $10,000 per year for the next two years. Assume your tuition and residence fees must be paid in equal instalments at the start of each semester (assume July 1 and January 1 as the semester-payment due dates). What is the timeline of your tuition and residence fee payments?

Solution

Assuming today is July 1 and your first payment occurs at date 0 (today). The remaining payments occur at semester-payment due dates. Using one semester as the period length, we can construct a timeline as follows:



CONCEPT CHECK

1. What are the key elements of a timeline?
2. How can you distinguish cash inflows from cash outflows on a timeline?

4.2 THE THREE RULES OF TIME TRAVEL

Financial decisions often require comparing or combining cash flows that occur at different points in time. In this section, we introduce three important rules central to financial decision making that allow us to compare or combine values.

RULE 1: ONLY CASH FLOW VALUES AT THE SAME POINT IN TIME CAN BE COMPARED OR COMBINED

Our first rule is that it is only possible to compare or combine values at the same point in time. This rule restates a conclusion introduced in Chapter 3: Only cash flows in the same units can be compared or combined. *A dollar today* and *a dollar in one year* are not equivalent. Having money now is more valuable than having money in the future; if you have the money today you can earn interest on it.

To compare or combine cash flows that occur at different points in time, you first need to convert the cash flows into the same units or *move* them to the same point in time. The next two rules show how to move the cash flows on the timeline.

RULE 2: TO MOVE A CASH FLOW FORWARD IN TIME, YOU MUST COMPOUND IT

Suppose we have \$1000 today, and we wish to determine the equivalent amount in one year's time. If the current market interest rate is 10%, we can use that rate as an exchange rate to move the cash flow forward in time. That is,

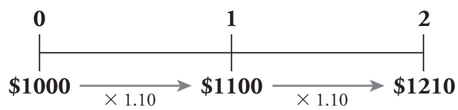
$$(\$1000 \text{ today}) \times (1.10 \text{ \$ in one year/\$ today}) = \$1100 \text{ in one year}$$

In general, if the market interest rate for the year is r, then we multiply by the **interest rate factor**, $(1 + r)$, to move the cash flow from the beginning to the end of the year. This process of moving a value or cash flow forward in time is known as **compounding**. *Our second rule stipulates that to move a cash flow forward in time, you must compound it.*

We can apply this rule repeatedly. Suppose we want to know how much the \$1000 is worth in two years' time. If the interest rate for year 2 is also 10%, then we convert as we just did:

$$(\$1100 \text{ in one year}) \times (1.10 \text{ \$ in two years/\$ in one year}) = \$1210 \text{ in two years}$$

Let's represent this calculation on a timeline:



Given a 10% interest rate, all of the cash flows—\$1000 at date 0, \$1100 at date 1, and \$1210 at date 2—are equivalent. They have the same value but are expressed in different units (different points in time). An arrow that points to the right indicates that the value is being moved forward in time, that is, compounded.

The value of a cash flow that is moved forward in time is known as its **future value**. In the preceding example, \$1210 is the future value of \$1000 two years from today. Note that the value grows as we move the cash flow further in the future. The equivalent value of two cash flows at two different points in time is sometimes referred to as the **time value of money**. By having money sooner, you can invest it and end up with more money later. Note also that the equivalent value grows by \$100 the first year, but by \$110 the second year. In the second year we earn interest on our original \$1000 principal, plus we earn interest on the \$100 accrued interest from the first year. If an investment only earns interest on principal and no interest on accrued interest, it is said to earn **simple interest**. Most investments earn interest on the original principal amount invested and earn interest on the accrued interest; the combined effect is known as **compound interest**.

How does the future value change if we move the cash flow to three years? Continuing to use the same approach, we compound the cash flow a third time. Assuming the competitive market interest rate is fixed at 10%, we get

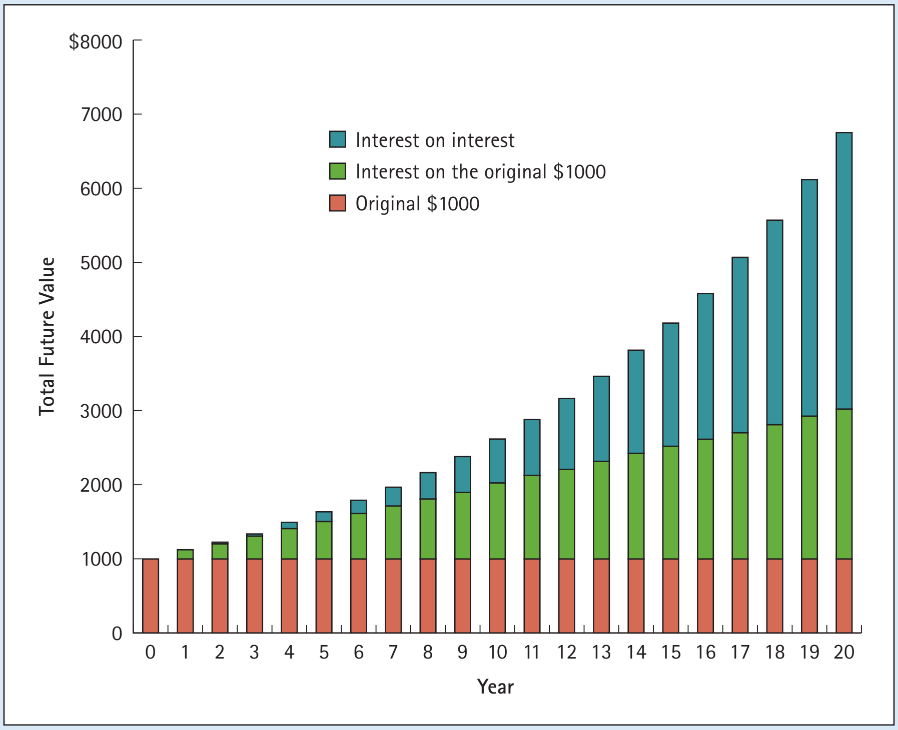
$$\$1000 \times (1.10) \times (1.10) \times (1.10) = \$1000 \times (1.10)^3 = \$1331$$

In general, if we have a cash flow now, C_0, to compute its value n periods into the future, we must compound it by the n intervening interest rate factors. If the interest rate r is constant, this calculation yields

Future Value of a Cash Flow

$$FV_n = C_0 \times \underbrace{(1 + r) \times (1 + r) \times \cdots \times (1 + r)}_{n \text{ times}} = C_0(1 + r)^n \qquad (4.1)$$

FIGURE 4.1



The Composition of Interest over Time

This bar graph shows how the account balance and the composition of the interest changes over time when an investor starts with an original deposit of $1000, represented by the red area, in an account earning 10% interest over a 20-year period. Note that the turquoise area representing interest on interest grows, and by year 15 has become larger than the interest on the original deposit, shown in green. Over the 20 years of the investment, the interest on interest the investor earned is $3727.50, while the interest earned on the original $1000 principal is $2000. The total compound interest over the 20 years is $5727.50 (the sum of the interest on interest and the interest on principal). Combining the original principal of $1000 with the total compound interest gives the future value after 20 years of $6727.50.

Figure 4.1 shows the importance of earning "interest on interest" in the growth of the account balance over time. The type of growth that results from compounding is called geometric or exponential growth. As Example 4.2 shows, over a long horizon, the effect of compounding can be quite dramatic.

EXAMPLE 4.2 THE POWER OF COMPOUNDING

Problem

Suppose you invest $1000 in an account paying 10% interest per year. How much will you have in the account in 7 years? in 20 years? in 75 years?

Solution

We can apply Eq. 4.1 to calculate the future value in each case:

$$\begin{aligned} &\text{7 years:} && \$1000(1.10)^{7} = \$1948.72 \\ &\text{20 years:} && \$1000(1.10)^{20} = \$6727.50 \\ &\text{75 years:} && \$1000(1.10)^{75} = \$1{,}271{,}895.37 \end{aligned}$$

Note that at 10% interest, our money will nearly double in 7 years. After 20 years, it will increase almost 7-fold. And if we invest for 75 years, we will be millionaires!

RULE OF 72

Another way to think about the effect of compounding and discounting is to consider how long it will take your money to double given different interest rates. Suppose we want to know how many years it will take for $1 to grow to a future value of $2. We want the number of years, n, to solve

$$FV_n = \$1 \times (1 + r)^n = \$2$$

If you solve this formula for different interest rates, you will find the following approximation:

$$\text{Years to Double} \approx 72 \div (\text{Interest Rate in Percent})$$

This simple "Rule of 72" is fairly accurate (i.e., within one year of the exact doubling time) for interest rates higher than 2%. For example, if the interest rate is 9%, the doubling time should be about 72 ÷ 9 = 8 years. Indeed, $1.09^8 = 1.99$! So, given a 9% interest rate, your money will approximately double every 8 years.

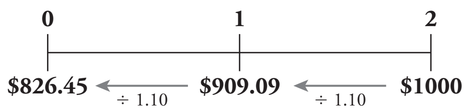
RULE 3: TO MOVE A CASH FLOW BACKWARD IN TIME, YOU MUST DISCOUNT IT

The third rule describes how to move cash flows backward in time. Suppose you would like to compute the value today of $1000 you anticipate receiving in one year. If the current market interest rate is 10%, you can compute this value by converting units as we did in Chapter 3:

$$(\$1000 \text{ in one year}) \div (1.10 \text{ \$ in one year/\$ today}) = \$909.09 \text{ today}$$

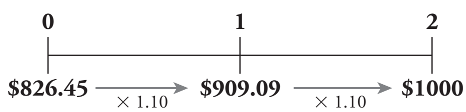
That is, to move the cash flow backward in time, we divide it by the interest rate factor, $(1 + r)$, where r is the interest rate – this is the same as multiplying by the **discount factor**, $\frac{1}{(1+r)}$. This process of moving a value or cash flow backward in time—finding the equivalent value today of a future cash flow—is known as **discounting**. *Our third rule stipulates that to move a cash flow back in time, we must discount it.*

To illustrate, suppose that you anticipate receiving the $1000 two years from today rather than in one year. If the interest rate for both years is 10%, we can prepare the following timeline:

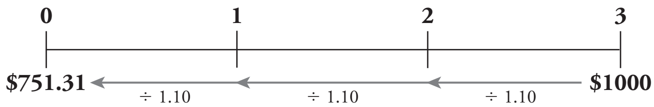


When the interest rate is 10%, all of the cash flows—$826.45 at date 0, $909.09 at date 1, and $1000 at date 2—are equivalent. They represent the same value in different units (different points in time). The arrow points to the left to indicate that the value is being moved backward in time or discounted. Note that the value decreases as we move the cash flow further back.

The value of a future cash flow at an earlier point on the timeline is its present value at the earlier point in time. That is, $826.45 is the present value at date 0 of $1000 in two years. Recall from Chapter 3 that the present value is the "do-it-yourself" price to produce a future cash flow. Thus, if we invested $826.45 today for two years at 10% interest, we would have a future value of $1000, using the second rule of time travel:



Suppose the \$1000 were three years away and you wanted to compute the present value. Again, if the interest rate is 10%, we have



That is, the present value today of a cash flow of \$1000 in three years is given by

$$\$1000 \div (1.10) \div (1.10) \div (1.10) = \$1000 \div (1.10)^3 = \$751.31$$

In general, to compute the present value today (date 0) of a cash flow C_n that comes n periods from now, we must discount it by the n intervening interest rate factors. If the interest rate r is constant, this yields

Present Value of a Cash Flow

$$PV_0 = C_n \div (1 + r)^n = \frac{C_n}{(1 + r)^n} \tag{4.2}$$

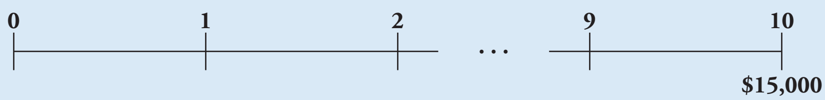
EXAMPLE 4.3 PRESENT VALUE OF A SINGLE FUTURE CASH FLOW

Problem

You are considering investing in a savings bond that will pay \$15,000 in 10 years. If the competitive market interest rate is fixed at 6% per year, what is the bond worth today?

Solution

The cash flows for this bond are represented by the following timeline:



Thus, the bond is worth \$15,000 in 10 years. To determine the value today, we compute the present value:

$$PV_0 = \frac{\$15{,}000}{1.06^{10}} = \$8375.92 \text{ today}$$

The bond is worth much less today than its final payoff because of the time value of money.

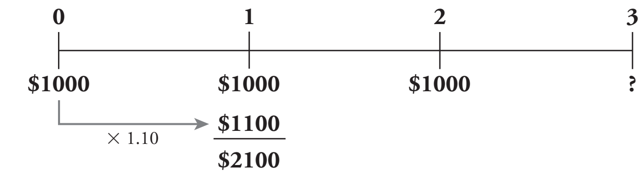
APPLYING THE RULES OF TIME TRAVEL

The rules of time travel allow us to compare and combine cash flows that occur at different points in time. Suppose we plan to save \$1000 today, and \$1000 at the end of each of the next two years. If we earn a fixed 10% interest rate on our savings, how much will we have three years from today?

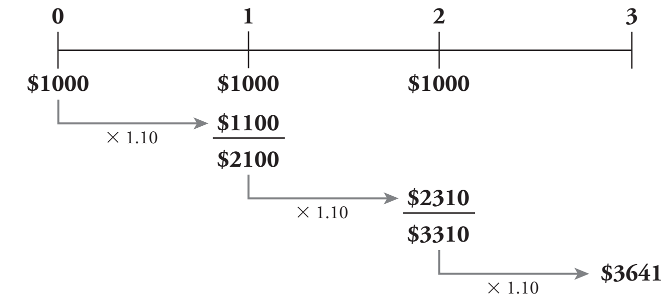
Again, we start with a timeline:

The timeline shows the three deposits we plan to make. We need to compute their value at the end of three years.

We can use the rules of time travel in a number of ways to solve this problem. First, we can take the deposit at date 0 and move it forward to date 1. Because it is then in the same time period as the date 1 deposit, we can combine the two amounts to find out the total in the bank on date 1:

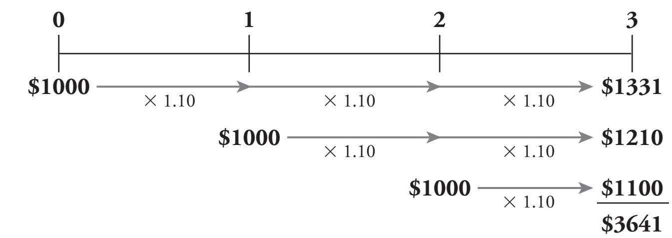


Using the first two rules of time travel, we find that our total savings on date 1 will be $2100. Continuing in this fashion, we can solve the problem as follows:



The total amount we will have in the bank at the end of three years is $3641. This amount is the future value of our $1000 savings deposits.

Another approach to the problem is to compute the future value in year 3 of each cash flow separately. Once all three amounts are in year 3 dollars, we can then combine them.



Both calculations give the same future value. As long as we follow the rules, we get the same result. The order in which we apply the rules does not matter. The calculation we choose depends on which is more convenient for the problem at hand. Table 4.1 summarizes the three rules of time travel and their associated formulas.

THE THREE RULES OF TIME TRAVEL

TABLE 4.1

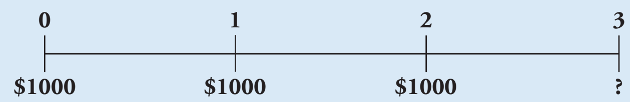
Rule 1	Only cash flow values at the same point in time can be compared or combined.	
Rule 2	To move a cash flow forward in time n periods, you must compound it.	Future Value of a Cash Flow $FV_n = C_0 \times (1 + r)^n$
Rule 3	To move a cash flow backward in time n periods, you must discount it.	Present Value of a Cash Flow $PV_0 = \dfrac{C_n}{(1 + r)^n}$

EXAMPLE 4.4 COMPUTING THE FUTURE VALUE

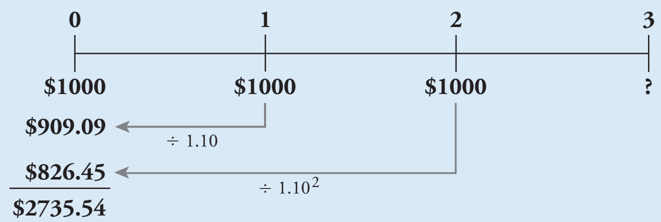
Problem

Let's revisit the savings plan we considered earlier: We plan to save $1000 today and at the end of each of the next two years. At a fixed 10% interest rate, how much will we have in the bank three years from today?

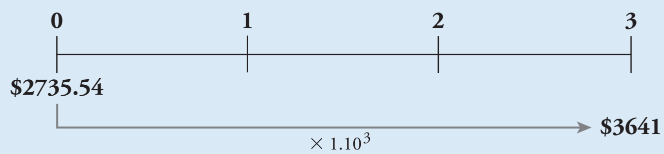
Solution



Let's solve this problem in a different way than we did earlier. First compute the present value of the cash flows. There are several ways to perform this calculation. Here we treat each cash flow separately and then combine the present values.



Saving $2735.54 today is equivalent to saving $1000 per year for three years. Now let's compute its future value in year 3:



This answer of $3641 is precisely the same result we found earlier. As long as we apply the three rules of time travel, we will always get the correct answer.

CONCEPT CHECK

1. Can you compare or combine cash flow values that are at different points in time?
2. How do you move a cash flow backward and forward in time?
3. What is compound interest?
4. Why does the future value of an investment grow faster in later years as shown in Figure 4.1?

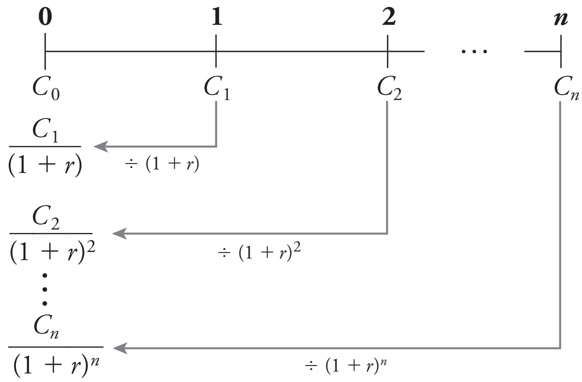
4.3 VALUING A STREAM OF CASH FLOWS

Most investment opportunities have multiple cash flows that occur at different points in time. In Section 4.2, we applied the rules of time travel to value such cash flows. Now we formalize this approach by deriving a general formula for valuing a stream of cash flows.

Consider a stream of cash flows: C_0 at date 0, C_1 at date 1, and so on, up to C_n at date n. We represent this cash flow stream on a timeline as follows:

Using the time travel techniques, we compute the present value of this cash flow stream in two steps. First, we compute the present value of each individual cash flow. Then, once the cash flows are in common units of dollars today, we can combine them.

For a given interest rate r, we represent this process on the timeline as follows:



Combining the dollar-today amounts we get $C_0 + \dfrac{C_1}{(1+r)} + \dfrac{C_2}{(1+r)^2} + \cdots + \dfrac{C_n}{(1+r)^n}$

This timeline provides the general formula for the present value today, at date 0, of a cash flow stream:

$$PV_0 = C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_n}{(1+r)^n} = \sum_{t=0}^{n} \frac{C_t}{(1+r)^t} \tag{4.3}$$

The summation sign, Σ, means "sum the individual elements for each date t from 0 to n." Note that $(1+r)^0 = 1$, so this shorthand matches precisely the long form of the equation. That is, the present value today of the cash flow stream is the sum of the present values of each cash flow. Recall from Chapter 3 how we defined the present value as the dollar amount you would need to invest today to produce the single cash flow in the future. The same idea holds in this context. The present value is the amount you need to invest today to generate the cash flow stream $C_0, C_1, \ldots C_n$. That is, receiving those cash flows is equivalent to having their present value in the bank today.

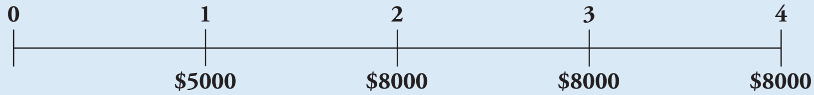
EXAMPLE 4.5 PRESENT VALUE OF A STREAM OF CASH FLOWS

Problem

You have just graduated and need money to buy a new car. Your rich Uncle Henry will lend you the money so long as you agree to pay him back within four years, and you offer to pay him the rate of interest that he would otherwise get by putting his money in a savings account. Based on your earnings and living expenses, you think you will be able to pay him \$5000 in one year, and then \$8000 each year for the next three years. If Uncle Henry would otherwise earn 6% per year on his savings, how much can you borrow from him?

Solution

The cash flows you can promise Uncle Henry are as follows:



How much money should Uncle Henry be willing to give you today in return for your promise of these payments? He should be willing to give you an amount that is equivalent to these payments in present value terms. This is the amount of money that it would take him to produce these same cash flows, which we calculate as follows:

$$
\begin{aligned}
PV_0 &= \frac{\$5000}{1.06} + \frac{\$8000}{1.06^2} + \frac{\$8000}{1.06^3} + \frac{\$8000}{1.06^4} \\
&= \$4716.98 + \$7119.97 + \$6716.95 + \$6336.75 \\
&= \$24{,}890.65
\end{aligned}
$$

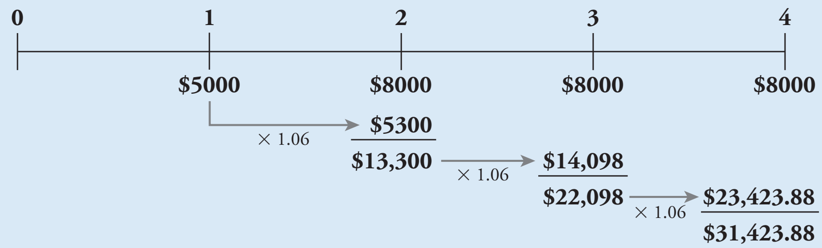
Thus, Uncle Henry should be willing to lend you \$24,890.65 in exchange for your promised payments. This amount is less than the total you will pay him, \$5000 + \$8000 + \$8000 + \$8000 = \$29,000, due to the time value of money.

Let's verify our answer. If your uncle kept his \$24,890.65 in the bank today earning 6% interest, in four years he would have

$$FV_4 = \$24{,}890.65 \times (1.06)^4 = \$31{,}423.87 \text{ in 4 years}$$

Now suppose that Uncle Henry gives you the money, and then deposits your payments to him in the bank each year. How much will he have four years from now?

We need to compute the future value of the annual deposits. One way to do so is to compute the bank balance each year:



We get the same answer both ways (within a penny, which is because of rounding).

The last section of Example 4.5 illustrates a general point. If you want to compute the future value of a stream of cash flows, you can do it directly (the second approach used in Example 4.5), or you can first compute the present value and then move it to the future (the first approach). Because we obey the laws of time travel in both cases, we get the same result. This principle can be applied more generally to write the following formula for the future value in year n in terms of the present value of a set of cash flows:

Future Value of a Cash Flow Stream with a Present Value of *PV*

$$FV_n = PV_0 \times (1 + r)^n \tag{4.4}$$

CONCEPT CHECK

1. How do you calculate the present value of a cash flow stream?
2. How do you calculate the future value of a cash flow stream?

4.4 CALCULATING THE NET PRESENT VALUE

Now that we have established the rules of time travel and determined how to compute present and future values, we are ready to address our central goal: calculating the *NPV* of future cash flows to evaluate an investment decision. Recall from Chapter 3 that we defined the net present value (*NPV*) of an investment decision as follows:

$$NPV = PV(\text{benefits}) - PV(\text{costs})$$

In this context, the benefits are the cash inflows and the costs are the cash outflows. We can represent any investment decision on a timeline as a cash flow stream where the cash outflows (investments) are negative cash flows and the inflows are positive cash flows. Thus, the *NPV* of an investment opportunity is also the *present value* of the stream of cash flows of the opportunity:

$$NPV = PV(\text{benefits}) - PV(\text{costs}) = PV(\text{benefits} - \text{costs})$$

EXAMPLE 4.6 *NPV* OF AN INVESTMENT OPPORTUNITY

Problem

You have been offered the following investment opportunity: If you invest \$1000 today, you will receive \$500 at the end of each of the next three years. If you could otherwise earn 10% per year on your money, should you undertake the investment opportunity?

Solution

As always, start with a timeline. We denote the upfront investment as a negative cash flow (because it is money we need to spend) and the money we receive as a positive cash flow.

0	**1**	**2**	**3**
−\$1000	**\$500**	**\$500**	**\$500**

To decide whether we should accept this opportunity, we compute the *NPV* by computing the present value of the stream:

$$NPV = -\$1000 + \frac{\$500}{1.10} + \frac{\$500}{1.10^2} + \frac{\$500}{1.10^3} = \$243.43$$

Because the *NPV* is positive, the benefits exceed the costs and we should make the investment. Indeed, the *NPV* tells us that taking this opportunity is like getting an extra \$243.43 that you can spend today. To illustrate, suppose you borrow \$1000 to invest in the opportunity and an extra \$243.43 to spend today. How much would you owe on the \$1243.43 loan in three years? At 10% interest, the amount you would owe would be

$$FV_3 = (\$1000 + \$243.43) \times (1.10)^3 = \$1655 \text{ in 3 years}$$

At the same time, the investment opportunity generates cash flows. If you put these cash flows into a bank account, how much will you have saved three years from now? The future value of the savings is

$$FV_3 = (\$500 \times 1.10^2) + (\$500 \times 1.10) + \$500 = \$1655 \text{ in 3 years}$$

As you see, you can use your bank savings to repay the loan. Taking the opportunity therefore allows you to spend \$243.43 today at no extra cost. The *NPV* actually shows how your wealth increases today if you accept the investment opportunity!

CALCULATING PRESENT VALUES IN EXCEL

While present and future value calculations can be done with a calculator, it is often convenient to evaluate them using a spreadsheet. In fact, we have used spreadsheets for most of the calculations in this book. A major advantage of using spreadsheets is that you can display numbers showing a specified number of decimal places (e.g., showing dollar amounts with two decimal places for the cents) but the spreadsheet keeps the full precision non-rounded numbers and thus rounding errors do not cumulate to cause error in the final solution. For example, the following spreadsheet calculates the *NPV* in Example 4.6:

	A	B	C	D	E
1	Discount Rate	10.0%			
2	Period	0	1	2	3
3	Cash Flow C_t	(1,000.0)	500.0	500.0	500.0
4	Discount Factor	1.000	0.909	0.826	0.751
5	PV(C_t)	(1,000.0)	454.5	413.2	375.7
6	**NPV**	**243.43**			

Rows 1–3 provide the key data of the problem, the discount rate and the cash flow timeline. Row 4 then calculates the discount factor, $1/(1 + r)^n$, that we use to convert the cash flow to its present value, shown in row 5. Finally, row 6 shows the sum of the present values of all the cash flows, which is the *NPV*. The formulas in rows 4–6 are shown below:

	A	B	C	D	E
4	Discount Factor	=1/(1+$B$1)^B2	=1/(1+$B$1)^C2	=1/(1+$B$1)^D2	=1/(1+$B$1)^E2
5	PV(C_t)	=B3*B4	=C3*C4	=D3*D4	=E3*E4
6	**NPV**	**=SUM(B5:E5)**			

Alternatively, we could have computed the entire *NPV* in one step, using a single (long) formula. We recommend as a best practice that you avoid that temptation and calculate the *NPV* step by step. Doing so facilitates error checking and makes clear the contribution of each cash flow to the overall *NPV*.

Excel's *NPV* Function

Excel also has a built-in *NPV* function. This function has the format, *NPV*(rate, value1, value2, ...) where "rate" is the interest rate per period used to discount the cash flows, and "value1", "value2", and so on are the cash flows (or ranges of cash flows). Unfortunately, however, the *NPV* function computes the present value of the cash flows *assuming the first cash flow occurs at date 1*. Therefore, if a project's first cash flow occurs at date 0, we must add it separately. For example, in the spreadsheet above, we would need the formula = B3 + *NPV*(B1, C3:E3) to calculate the *NPV* of the indicated cash flows.

Another pitfall with the *NPV* function is that cash flows that are left blank are treated differently from cash flows that are equal to zero. If the cash flow is left blank, *both the cash flow and the period are ignored*. For example, consider the example below in which the year 2 cash flow has been deleted:

	A	B	C	D	E
1	Discount Rate	10.0%			
2	Period	0	1	2	3
3	Cash Flow C_t	(1,000.0)	500.0		500.0
4	Discount Factor	1.000	0.909	0.826	0.751
5	PV(C_t)	(1,000.0)	454.5	-	375.7
6	**NPV**	**(169.80)**	=SUM(B5:E5)		
7	NPV function	(132.23)	=B3+NPV(B1,C3:E3)		

Our original method provides the correct solution in row 6, whereas the *NPV* function used in row 7 treats the cash flow on date 3 as though it occurred at date 2, which is clearly not what is intended and is incorrect.

In principle, we have explained how to answer the question we posed at the beginning of the chapter: How should financial managers evaluate a project such as undertaking the development of the 787 Dreamliner airplane? We have shown how to compute the *NPV* of an investment opportunity such as the 787 Dreamliner airplane that lasts more than one period. In practice, when the number of cash flows exceeds four or five (as it most likely will), the calculations can become tedious. Fortunately, a number of special cases do not require us to treat each cash flow separately. We derive these shortcuts in the next section.

CONCEPT CHECK

1. How do you calculate the *NPV* of a cash flow stream?
2. What benefit does a firm receive when it accepts a project with a positive *NPV*?

4.5 PERPETUITIES AND ANNUITIES

The formulas we have developed so far allow us to compute the present or future value of any cash flow stream. In this section, we consider two types of assets, *perpetuities* and *annuities*, and learn shortcuts for valuing them. These shortcuts are possible because the cash flows follow a regular pattern.

REGULAR PERPETUITIES

A **regular perpetuity** is a stream of equal cash flows that occur at constant time intervals and last forever. Often a regular perpetuity will simply be referred to as a perpetuity (later in the chapter we also discuss growing perpetuities). One example of a regular perpetuity is the British government bond called a **consol** (or perpetual bond). Consol bonds promise the owner a fixed cash flow every year, forever.

Here is the timeline for a perpetuity:

Note from the timeline that the first cash flow does not occur immediately; *it arrives at the end of the first period*. This timing is sometimes referred to as payment *in arrears* and is a standard convention that we adopt throughout this text.

Using the formula for the present value, the present value today of a perpetuity with payment C and interest rate r is given by

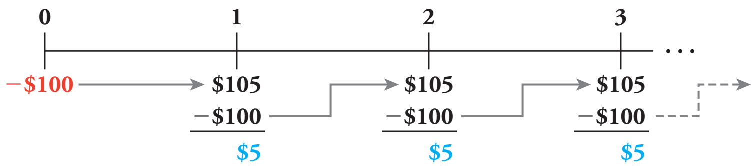
$$PV_0 = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t}$$

Notice that $C_t = C$ in the present value formula because the cash flow for a perpetuity is constant. Also, because the first cash flow is in one period, $C_0 = 0$.

To find the value of a perpetuity one cash flow at a time would take forever—literally! You might wonder how, even with a shortcut, the sum of an infinite number of positive terms could be finite. The answer is that the cash flows in the future are discounted for an ever increasing number of periods, so their contribution to the sum eventually becomes negligible.[2]

2. In mathematical terms, this is a geometric series, so it converges if $r > 0$.

To derive the shortcut, we calculate the value of a perpetuity by creating our own perpetuity. We can then calculate the present value of the perpetuity because, by the Law of One Price, the value of the perpetuity must be the same as the cost we incurred to create our own perpetuity. To illustrate, suppose you could invest \$100 in a bank account paying 5% interest per year forever. At the end of one year, you will have \$105 in the bank: your original \$100 plus \$5 in interest. Suppose you withdraw the \$5 interest and reinvest the \$100 for a second year. Again you will have \$105 after one year, and you can withdraw \$5 and reinvest \$100 for another year. By doing this year after year, you can withdraw \$5 every year in perpetuity:



By investing \$100 in the bank today, you can, in effect, create a perpetuity paying \$5 per year (we are assuming the bank will remain solvent and the interest rate will not change). Recall from Chapter 3 that the Law of One Price tells us that the same good must have the same price in every market. Because the bank will "sell" us (allow us to create) the perpetuity for \$100, the present value of the \$5 per year in perpetuity is this "do-it-yourself" cost of \$100.

Now let's generalize this argument. Suppose we invest an amount P in the bank. Every year we can withdraw the interest we have earned, $C = r \times P$, leaving the principal, P, in the bank. The present value of receiving C in perpetuity is therefore the upfront cost $P = C/r$. Therefore, we have the following equation:

Present Value Today (date 0) of a Perpetuity with Discount Rate, *r*, and Constant Cash Flows, *C*, Starting in One Period (date 1)

$$PV_0 = \frac{C}{r} \tag{4.5}$$

By depositing the amount $\frac{C}{r}$ today, we can withdraw interest of $\frac{C}{r} \times r = C$ each period in perpetuity.

HISTORICAL EXAMPLES OF PERPETUITIES

Perpetual bonds were some of the first bonds ever issued. The oldest perpetuities that are still making interest payments were issued in 1648 by the Hoogheemraadschap Lekdijk Bovendams, a seventeenth-century Dutch water board responsible for upkeep of the local dikes. To verify that these bonds continue to pay interest, two finance professors at Yale University, William Goetzmann and Geert Rouwenhorst purchased one of these bonds in July 2003 and collected 26 years' back interest. On its issue date in 1648, this bond originally paid interest in Carolus guilders. Over the next 355 years, the currency of payment changed to Flemish pounds, Dutch guilders, and most recently euros. Currently, the bond pays interest of €11.34 annually.

Although the Dutch bonds are the oldest perpetuities still in existence, the first perpetuities date from much earlier times. For example, *cencus agreements* and *rentes*, which were forms of perpetuities and annuities, were issued in the twelfth century in Italy, France, and Spain. They were initially designed to circumvent the usury laws of the Catholic Church: Because they did not require the repayment of principal, in the eyes of the Church they were not considered loans.

Note the logic of our argument. To determine the present value of a cash flow stream, we computed the "do-it-yourself" cost of creating those same cash flows at the bank. This is an extremely useful and powerful approach—and is much simpler and faster than summing those infinite terms![3]

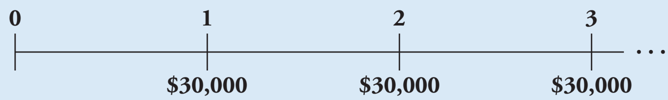
EXAMPLE 4.7 ENDOWING A PERPETUITY

Problem

You want to endow an annual graduation party at your university. You want the event to be a memorable one, so you budget \$30,000 per year forever for the party. If the university earns 8% per year on its investments, and if the first party is in one year's time, how much will you need to donate to endow the party?

Solution

The timeline of the cash flows you want to provide is



This is a standard perpetuity of \$30,000 per year. The funding you would need to give the university in perpetuity is the present value of this cash flow stream. From the formula,

$$PV_0 = C/r = \$30{,}000/0.08 = \$375{,}000 \text{ today}$$

if you donate \$375,000 today, and if the university invests it at 8% per year forever, then the graduates will have \$30,000 every year for their party. Hopefully, they will invite you back to attend!

ANNUITIES

A **regular annuity** is a stream of n equal cash flows paid over constant time intervals. As with regular perpetuities, we often call regular annuities simply as annuities (and we will introduce growing annuities later in the chapter). The difference between an annuity and a perpetuity is that an annuity ends after some fixed number of payments. Most car loans, mortgages, and some bonds are annuities. We represent the cash flows of an annuity on a timeline as follows.

Note that just as with the perpetuity, we adopt the convention that the first payment takes place at date 1, one period from today. The present value of an n-period annuity with payment C and interest rate r is

$$PV_0 = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^n} = \sum_{t=1}^{n} \frac{C}{(1+r)^t}$$

3. Another mathematical derivation of this result exists (see the online appendix), but it is less intuitive. This case is a good example of how the Law of One Price can be used to derive useful results.

COMMON MISTAKE DISCOUNTING ONE TOO MANY TIMES

The perpetuity formula assumes that the first payment occurs at the end of the first period (at date 1). Sometimes perpetuities have cash flows that start later in the future. In this case, we can adapt the perpetuity formula to compute the present value, but we need to do so carefully to avoid a common mistake.

To illustrate, consider the graduation party described in Example 4.6. Rather than starting immediately, suppose that the first party will be held two years from today. How would this delay change the amount of the donation required?

Now the timeline looks like this:

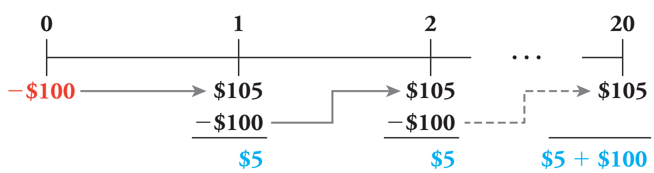
We need to determine the present value of these cash flows, as it tells us the amount of money in the bank needed today to finance the future parties. We cannot apply the perpetuity formula directly, however, because these cash flows are not *exactly* a perpetuity as we defined it. Specifically, the cash flow in the first period is "missing." But consider the situation on date 1—at that point, the first party is one period away and then the cash flows are periodic. From the perspective of date 1, this *is* a perpetuity, and we can apply the formula. From the preceding calculation, we know we need $PV_1 = \$375{,}000$ on date 1 to have enough to start the parties on date 2. We rewrite the timeline as follows:

Our goal can now be restated more simply: How much do we need to invest today to have $375,000 in one year? This is a simple present value calculation:

$$PV_0 = \$375{,}000/1.08 = \$347{,}222.22 \text{ today}$$

A common mistake is to discount the $375,000 twice because the first party is in two periods. *Remember—the present value formula for the perpetuity already discounts the cash flows to one period prior to the first cash flow.* Note that the length of the period is determined by the time period between cash flows. In the above example, the present value for the perpetuity brings the cash flows back one year before the first cash flow because the cash flows and interest rate are yearly. If a perpetuity had monthly cash flows (and we used the appropriate monthly discount rate), then the present value formula for the perpetuity would bring the cash flows back one month before the first cash flow. Keep in mind that this applies to perpetuities, annuities, and all of the other special cases discussed in this section. All of these formulas discount the cash flows to one period prior to the first cash flow.

To find a simpler formula, we use the same approach we followed with the perpetuity: Find a way to create an annuity. To illustrate, suppose you invest $100 in a bank account paying 5% interest. At the end of one year, you will have $105 in the bank—your original $100 plus $5 in interest. Using the same strategy as for a perpetuity, suppose you withdraw the $5 interest and reinvest the $100 for a second year. Once again you will have $105 after one year, and you can repeat the process, withdrawing $5 and reinvesting $100, every year. For a perpetuity, you left the principal in forever. Alternatively, you might decide after 20 years to close the account and withdraw the principal. In that case, your cash flows will look like this:



With your initial \$100 investment, you have created a 20-year annuity of \$5 per year, plus you will receive an extra \$100 at the end of 20 years. By the Law of One Price, because it took an initial investment of \$100 to create the cash flows on the timeline, the present value of these cash flows is \$100, or

$$\$100 = PV(\text{20-year annuity of \$5 per year}) + PV(\text{\$100 in 20 years})$$

Rearranging terms gives

$$\begin{aligned} PV(\text{20-year annuity of \$5 per year}) &= \$100 - PV(\text{\$100 in 20 years}) \\ &= \$100 - \frac{\$100}{(1.05)^{20}} = \$62.31 \end{aligned}$$

So the present value of \$5 for 20 years is \$62.31. Intuitively, the value of the annuity is the initial investment in the bank account minus the present value of the principal that will be left in the account after 20 years.

We can use the same idea to derive the general formula. First, we invest P in the bank, and withdraw only the interest $C = r \times P$ each period. After n periods, we close the account. Thus, for an initial investment of P, we will receive an n-period annuity of C per period, *plus* we will get back our original P at the end. P is the total present value of the two sets of cash flows,[4] or

$$P = PV(\text{annuity of } C \text{ for } n \text{ periods}) + PV(P \text{ in period } n)$$

By rearranging terms, we compute the present value of the annuity:

$$\begin{aligned} PV(\text{annuity of } C \text{ for } n \text{ periods}) &= P - PV(P \text{ in period } n) \\ &= P - \frac{P}{(1+r)^n} = P\left(1 - \frac{1}{(1+r)^n}\right) \end{aligned} \tag{4.6}$$

Recall that the periodic payment C is the interest earned every period; that is, $C = r \times P$ or, equivalently, solving for P provides the upfront cost in terms of C,

$$P = \frac{C}{r}$$

Making this substitution for P, in Eq. 4.6, provides the formula for the present value of an annuity of C for n periods.[5]

Present Value Today (date 0) of an n-Period Annuity with Discount Rate, r, and Constant Cash Flows, C, Starting in One Period (date 1)

$$PV_0 = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) \tag{4.7}$$

4. Here we are using value additivity (see Chapter 3) to separate the present value of the cash flows into separate pieces.

5. An early derivation of this formula is attributed to the astronomer Edmond Halley ("Of Compound Interest," published after Halley's death by Henry Sherwin, *Sherwin's Mathematical Tables*, London: W. and J. Mount, T. Page and Son, 1761).

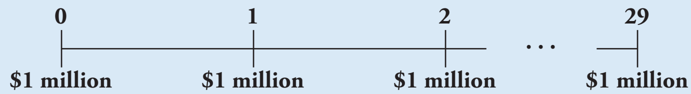
EXAMPLE 4.8 PRESENT VALUE OF A LOTTERY PRIZE ANNUITY

Problem

On a recent trip to the U.S., you purchased a ticket in a state lottery. Now you discover that you are the lucky winner of the $30 million prize. You can take your prize money as either (a) 30 payments of $1 million per year (starting today), or (b) $15 million paid today. If the interest rate is 8%, which option should you take?

Solution

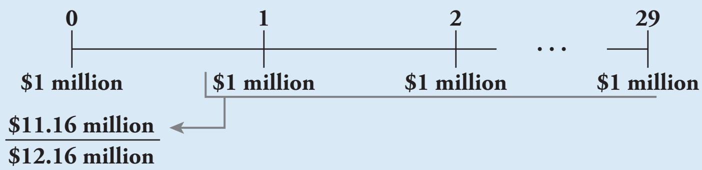
Option (a) provides $30 million in prize money but paid over time. To evaluate it correctly, we must convert it to a present value. Here is the timeline:



Because the first payment starts today, the last payment will occur in 29 years (for a total of 30 payments).[6] The $1 million at date 0 is already stated in present value terms, but we need to compute the present value of the remaining payments. Fortunately, this case looks like a 29-year annuity of $1 million per year, so we can use the annuity formula:

$$\begin{aligned} PV_0(\text{29-year annuity of \$1 million}) &= \$1{,}000{,}000 \times \frac{1}{0.08}\left(1 - \frac{1}{1.08^{29}}\right) \\ &= \$1{,}000{,}000 \times 11.15840601 \\ &= \$11{,}158{,}406.01 \text{ today} \end{aligned}$$

Thus, the total present value of the cash flows is $1,000,000 + $11,158,406.01 = $12,158,406.01. In timeline form:



Option (b), $15 million upfront, is more valuable—even though the total amount of money paid is half that of option (a). The reason for the difference is the time value of money. If you have the $15 million today, you can use $1 million immediately and invest the remaining $14 million at an 8% interest rate. This strategy will give you $14 million × 8% = $1.12 million per year in perpetuity! Alternatively, you can spend $15 million − $11.16 million = $3.84 million today, and invest the remaining $11.16 million, which will still allow you to withdraw $1 million each year for the next 29 years before your account is depleted.

Now that we have derived a simple formula for the present value of an annuity, it is easy to find a simple formula for the future value. If we want to know the value n years in the future, we move the present value n periods forward on the timeline; that is, we compound the present value for n periods at interest rate r:

6. An annuity in which the first payment occurs immediately is sometimes called an *annuity due*. Throughout this text, we always use the term "annuity" to mean one that is paid in arrears.

Future Value at Time of Last Payment of an *n*-Period Annuity with Discount Rate, *r*, and Constant Cash Flows, *C*

$$FV_n = PV_0 \times (1 + r)^n = \frac{C}{r}\left(1 - \frac{1}{(1 + r)^n}\right) \times (1 + r)^n$$

$$FV_n = C \times \frac{1}{r}\left((1 + r)^n - 1\right) \tag{4.8}$$

This formula is useful if we want to know how a savings account will grow over time.

EXAMPLE 4.9 REGISTERED RETIREMENT SAVINGS PLAN (RRSP) ANNUITY

Problem

Ellen just turned 35 years old, and she has decided it is time to plan seriously for her retirement. On each birthday, beginning in one year and ending when she turns 65, she will save \$10,000 in an RRSP account. If the account earns 10% per year, how much will Ellen have saved at age 65?

Solution

As always, we begin with a timeline. In this case, it is helpful to keep track of both the dates and Ellen's age:

Age:	*35*	*36*	*37*		*65*
Date:	**0**	**1**	**2**	. . .	**30**
		\$10,000	**\$10,000**		**\$10,000**

Ellen's RRSP looks like an annuity of \$10,000 per year for 30 years. (*Hint:* It is easy to become confused when you just look at age, rather than at both dates and age. A common error is to think there are only $65 - 36 = 29$ payments. Writing down both dates and age avoids this problem.)

To determine the amount Ellen will have in the RRSP at age 65, we compute the future value of this annuity:

$$\begin{aligned} FV_{30} &= \$10{,}000 \times \frac{1}{0.10}(1.10^{30} - 1) \\ &= \$10{,}000 \times 164.494023 \\ &= \$1{,}644{,}940.23 \text{ at age 65} \end{aligned}$$

GROWING CASH FLOWS

So far, we have considered only cash flow streams that have the same cash flow every period. If instead the cash flows are expected to grow at a constant rate in each period, we can also derive a simple formula for the present value of the future stream.

GROWING PERPETUITY. A **growing perpetuity** is a stream of cash flows that occur at regular intervals and grow at a constant rate forever. For example, a growing perpetuity with a first payment of \$100 that grows at a rate of 3% has the following timeline:

0	1	2	3	4	...
	\$100	**\$100 × 1.03 = \$103**	**\$103 × 1.03 = \$106.09**	**\$106.09 × 1.03 = \$109.27**	

In general, a growing perpetuity with a first payment C_1 and a growth rate g will have the following series of cash flows:

0	1	2	3	4	...
	C_1	$C_1 \times (1 + g)$	$C_1 \times (1 + g)^2$	$C_1 \times (1 + g)^3$	

As with perpetuities with equal cash flows, we adopt the convention that the first payment occurs at date 1. Since the first payment, C_1, occurs at date 1 and the t^{th} payment, C_t, occurs at date t, there are only $t - 1$ periods of growth between these payments. Substituting the cash flows from the preceding timeline into the general formula for the present value of a cash flow stream gives

$$PV_0 = \frac{C_1}{(1 + r)} + \frac{C_1(1 + g)}{(1 + r)^2} + \frac{C_1(1 + g)^2}{(1 + r)^3} + \cdots = \sum_{t=1}^{\infty} \frac{C_1(1 + g)^{t-1}}{(1 + r)^t}$$

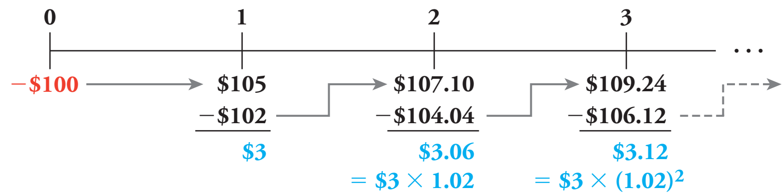
Suppose $g > r$. Then the cash flows grow even faster than they are discounted; each term in the sum gets larger, rather than smaller. In this case, the sum is infinite! What does an infinite present value mean? Remember that the present value is the "do-it-yourself" cost of creating the cash flows. An infinite present value means that no matter how much money you start with, it is *impossible* to reproduce those cash flows on your own. Growing perpetuities of this sort cannot exist in practice because no one would be willing to offer one at any finite price. A promise to pay an amount that forever grew faster than the interest rate is also unlikely to be kept (or believed by any savvy buyer).

The only viable growing perpetuities are those where the growth rate is less than the interest rate, so that each successive term in the sum is less than the previous term and the overall sum is finite. Consequently, we assume that $g < r$ for a growing perpetuity.

To derive the formula for the present value of a growing perpetuity, we follow the same logic used for a regular perpetuity: Compute the amount you would need to deposit today to create the perpetuity yourself. In the case of a regular perpetuity, we created a constant payment forever by withdrawing the interest earned each year and reinvesting the principal. To increase the amount we can withdraw each year, the principal that we reinvest each year must grow. We can accomplish this by withdrawing less than the full amount of interest earned each period, using the remaining interest to increase our principal.

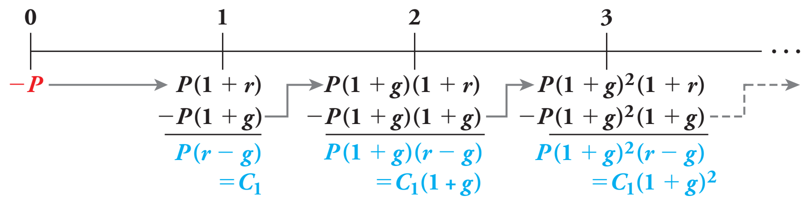
Let's consider a specific case. Suppose you want to create a perpetuity growing at 2%, so you invest \$100 in a bank account that pays 5% interest. At the end of one year, you will have \$105 in the bank—your original \$100 plus \$5 in interest. If you withdraw only \$3, you will have \$102 to reinvest—2% more than the amount you had initially.

This amount will then grow to \$102 × 1.05 = \$107.10 in the following year, and you can withdraw \$3 × 1.02 = \$3.06. which will leave you with principal of \$107.10 − \$3.06 = \$104.04. Note that \$102 × 1.02 = \$104.04. That is, both the amount you withdraw and the principal you reinvest grow by 2% each year. On a timeline, these cash flows look like this:



By following this strategy, you have created a growing perpetuity that starts at \$3 and grows 2% per year. This growing perpetuity must have a present value equal to the cost of \$100.

We can generalize this argument. In the case of an equal-payment perpetuity, we deposited an amount P in the bank and withdrew the interest each year. Because we always left the principal, P, in the bank, we could maintain this pattern forever. If we want to increase the amount we withdraw from the bank each year by g, then the principal in the bank will have to grow by the same factor g. That is, instead of reinvesting P in the second year, we should reinvest $P(1 + g) = P + gP$. In order to increase our principal by gP, we can only withdraw $C_1 = rP - gP = P(r - g)$.



From the timeline, we see that after one period we can withdraw $C_1 = P(r - g)$ and keep our account balance and cash flow growing at a rate of g forever. Solving this equation for P gives

$$P = \frac{C_1}{r - g}$$

The present value of the growing perpetuity with initial cash flow C_1 is P, the initial amount deposited in the bank account:

Present Value Today (date 0) of a Growing Perpetuity with Discount Rate, *r*, Growth Rate, *g*, and First Cash Flow, C_1, Starting in One Period (date 1)

$$PV_0 = \frac{C_1}{r - g} \tag{4.9}$$

To understand the formula for a growing perpetuity intuitively, start with the formula for a perpetuity. In the earlier case, you had to put enough money in the bank to ensure that the interest earned matched the cash flows of the regular perpetuity. In the case of a growing perpetuity, you need to put more than that amount in the bank because you have to finance the growth in the cash flows. How much more? If the bank pays interest at a rate of 10%, then all that is left to take out if you want to make sure the principal grows 3% per year is the difference: 10% − 3% = 7%. So instead of the present value of the perpetuity being the first cash flow divided by the interest rate, it is now the first cash flow divided by the *difference* between the interest rate and the growth rate.

EXAMPLE 4.10 ENDOWING A GROWING PERPETUITY

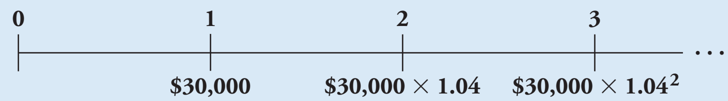
Problem

In Example 4.7, you planned to donate money to your university to fund an annual \$30,000 graduation party. Given an interest rate of 8% per year, the required donation was the present value of

$$PV_0 = \frac{\$30{,}000}{0.08} = \$375{,}000 \text{ today}$$

Before accepting the money, however, the president of the student association has asked that you increase the donation to account for the effect of inflation on the cost of the party in future years. Although \$30,000 is adequate for next year's party, the president estimates that the party's cost will rise by 4% per year thereafter. To satisfy the president's request, how much do you need to donate now?

Solution



The cost of the party next year is \$30,000, and the cost then increases 4% per year forever. From the timeline, we recognize the form of a growing perpetuity. To finance the growing cost, you need to provide the present value today of

$$PV_0 = \frac{\$30{,}000}{(0.08 - 0.04)} = \$750{,}000 \text{ today}$$

You need to double the size of your gift. Now you can be sure they will invite you to the future parties!

GROWING ANNUITY. A **growing annuity** is a stream of n growing cash flows, paid at regular intervals. It is a growing perpetuity that eventually comes to an end. The following timeline shows a growing annuity with initial cash flow C_1, growing at rate g every period until period n:

As with growing perpetuities discussed earlier, we adopt the convention that the first payment occurs at date 1. Since the first payment, C_1, occurs at date 1 and the n^{th} payment, C_n, occurs at date n, there are only $n - 1$ periods of growth between these payments.

The cash flows represented on the above timeline are equivalent to the cash flows of a growing perpetuity with same initial cash flow, C_1, and growth rate, g, but with all cash flows starting with C_{n+1} onward removed. The cash flows removed are simply a growing perpetuity with first cash flow of C_{n+1} that starts in date $n + 1$. Thus to determine the present value of the growing annuity with first cash flow, C_1, we can take the present value of a growing perpetuity with first cash flow, C_1, and subtract off the present value of a growing perpetuity with first cash flow, C_{n+1}, that starts at date $n + 1$. So we have the following:

$$PV_{0 \text{ of growing annuity}} = \underbrace{\frac{C_1}{r-g}}_{\substack{PV_0 \text{ of growing} \\ \text{perpetuity}}} - \underbrace{\underbrace{\frac{C_{n+1}}{r-g}}_{\substack{PV_n \text{ of growing} \\ \text{perpetuity that} \\ \text{starts at date} \\ n+1}} \times \frac{1}{(1+r)^n}}_{\substack{PV_0 \text{ of growing perpetuity that starts at} \\ \text{date } n+1}}$$

substituting in $C_1 \times (1+g)^n$ for C_{n+1} we get

$$PV_{0 \text{ of growing annuity}} = \frac{C_1}{r-g} - \frac{C_1 \times (1+g)^n}{r-g} \times \frac{1}{(1+r)^n}$$

Simplifying and collecting terms, the following formula results.

Present Value Today (date 0) of an *n*-Period Growing Annuity with Discount Rate *r*, Growth Rate *g*, and First Cash Flow C_1, Starting in One Period (date 1)

$$PV_0 = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right] \tag{4.10}$$

Because the annuity has only a finite number of terms, Eq. 4.10 also works when $g > r$.[7]

The formula for the present value of a growing annuity is a general solution. In fact, we can deduce all of the other formulas in this section from the expression for a growing annuity. To see how to derive the other formulas from this one, first consider a growing perpetuity. It is a growing annuity with $n = \infty$. If $g < r$, then

$$\frac{1+g}{1+r} < 1 \text{ and so } \lim_{n \to \infty}\left(\frac{1+g}{1+r}\right)^n = 0$$

So the formula for a growing annuity when $n = \infty$ therefore becomes

$$PV_0 = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right] = \frac{C_1}{r-g}(1-0) = \frac{C_1}{r-g}$$

which is the formula for a growing perpetuity. The formulas for the present values of a regular annuity and a perpetuity also follow from Eq. 4.10 if we let the growth rate, g, equal 0.

Similar to what we did with regular annuities, it is easy to find a simple formula for the future value of a growing annuity. If we want to know the value n years in the future, we move the present value n periods forward on the timeline; that is, we compound the present value for n periods at interest rate r:

$$PV_0 = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right]$$

$$FV_n = PV_0 \times (1+r)^n = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right] \times (1+r)^n$$

7. Eq. 4.10 does not work for $g = r$. But in that case, growth and discounting cancel out, and the present value, equivalent to receiving all the cash flows, is $PV_0 = n \times C_1/(1+r)$.

Multiplying through the brackets and simplifying, we get:

Future Value at Time of Last Payment of an *n*-Period Growing Annuity with Discount Rate, *r*, Growth Rate, *g*, and First Cash Flows, C_1

$$FV_n = \frac{C_1}{r - g}[(1 + r)^n - (1 + g)^n] \tag{4.11}$$

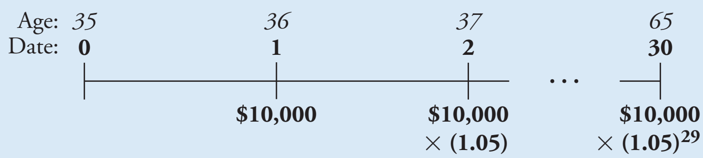
EXAMPLE 4.11 RETIREMENT SAVINGS WITH A GROWING ANNUITY

Problem

In Example 4.9, Ellen considered saving $10,000 per year for her retirement. Although $10,000 is the most she can save in the first year, she expects her salary to increase each year so that she will be able to increase her savings by 5% per year. With this plan, if she earns 10% per year in her RRSP, what is the present value of her planned savings and how much will Ellen have saved at age 65?

Solution

Her new savings plan is represented by the following timeline:



This example involves a 30-year growing annuity, with a growth rate of 5%, and an initial cash flow of $10,000. The present value of this growing annuity is given by

$$PV_0 = \frac{\$10{,}000}{0.10 - 0.05}\left[1 - \left(\frac{1.05}{1.10}\right)^{30}\right]$$
$$= \$150{,}463.15 \text{ today}$$

Ellen's proposed savings plan is equivalent to having $150,463.15 in the bank *today.* To determine the amount she will have at age 65, we could simply move this amount forward 30 years:

$$FV = \$150{,}463.15 \times 1.10^{30}$$
$$= \$2{,}625{,}491.98 \text{ in 30 years}$$

Alternatively, we could apply Eq. 4.11 to get the same amount:

$$FV_n = \frac{\$10{,}000}{0.10 - 0.05}[(1.10)^{30} - (1.05)^{30}] = \$2{,}625{,}491.98$$

Ellen will have saved about $2.625 million at age 65 using the new savings plan. This sum is almost $1 million more than she would have had in her RRSP without the additional annual increases in savings.

CONCEPT CHECK

1. How do you calculate the present value of
   a. a perpetuity?
   b. an annuity?
   c. a growing perpetuity?
   d. a growing annuity?

2. How are the formulas for the present value of a perpetuity, an annuity, a growing perpetuity, and a growing annuity related?
3. How do you calculate the future value of
   a. an annuity?
   b. a growing annuity?

4.6 SOLVING PROBLEMS WITH A SPREADSHEET

Spreadsheet software such as Excel and typical financial calculators have a set of functions that perform the calculations that finance professionals do most often. In Excel, the functions are called NPER, RATE, PV, PMT, and FV. The functions are all based on the timeline of an annuity:

together with an interest rate, denoted by *RATE*. Thus, there are a total of five variables: *NPER, RATE, PV, PMT,* and *FV*. Each function takes four of these variables as inputs and returns the value of the fifth one that ensures that the *NPV* of the cash flows is zero. That is, the functions all solve the problem

$$NPV = PV + PMT \times \frac{1}{RATE}\left(1 - \frac{1}{(1 + RATE)^{NPER}}\right) + \frac{FV}{(1 + RATE)^{NPER}} = 0 \tag{4.12}$$

In words, the present value of the annuity payments *PMT,* plus the present value of the final payment *FV,* plus the initial amount *PV,* has a net present value of zero. Let's tackle a few examples.

EXAMPLE 4.12

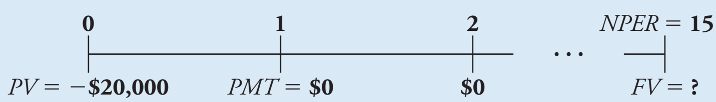
COMPUTING THE FUTURE VALUE IN EXCEL

Problem

Suppose you plan to invest \$20,000 in an account paying 8% interest. How much will you have in the account in 15 years?

Solution

We represent this problem with the following timeline:



To compute the solution, we enter the four variables we know $(NPER = 15, RATE = 8\%,$ $PV = -20{,}000, PMT = 0)$ and solve for the one we want to determine (*FV*) using the Excel function FV(RATE, NPER, PMT, PV). The spreadsheet here calculates a future value of \$63,443.38.

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	15	8.00%	−20,000	0		
Solve for FV					**63,443.38**	= FV(0.08, 15, 0, −20000)

Note that we entered *PV* as a negative number (the amount we are putting *into* the bank), and *FV* is shown as a positive number (the amount we can take *out* of the bank). It is important to use signs correctly to indicate the direction in which the money is flowing when using the spreadsheet functions or a financial calculator's finance functions.

To check the result, we can solve this problem directly:

$$FV_{15} = \$20{,}000 \times 1.08^{15} = \$63{,}443.38$$

This Excel spreadsheet in Example 4.12 is available on the MyFinanceLab Web site and is set up to allow you to compute any one of the five variables. We refer to this spreadsheet as the **annuity spreadsheet**. You simply enter the four input variables on the top line and leave the variable you want to compute blank. The spreadsheet computes the fifth variable and displays the answer on the bottom line. The spreadsheet also displays the Excel function that is used to get the answers. Let's work through a more complicated example, Example 4.13, that illustrates the convenience of the annuity spreadsheet.

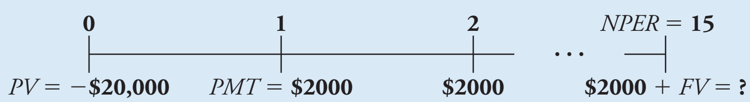
EXAMPLE 4.13 USING THE ANNUITY SPREADSHEET

Problem

Suppose that you invest \$20,000 in an account paying 8% interest. You plan to withdraw \$2000 at the end of each year for 15 years. How much money will be left in the account after 15 years?

Solution

Again, we start with the timeline:



The timeline indicates that the withdrawals are an annuity payment that we receive from the bank account. Note that *PV* is negative (money *into* the bank), while *PMT* is positive (money *out* of the bank). We solve for the final balance in the account, *FV*, using the annuity spreadsheet:

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	15	8.00%	−20,000	2000		
Solve for FV					**9139.15**	= FV(0.08, 15, 2000, −20000)

We will have \$9139.15 left in the bank after 15 years.

We can also compute this solution directly. One approach is to think of the deposit and the withdrawals as being separate accounts. In the account with the \$20,000 deposit, our savings will grow to \$63,443.38 in 15 years, as we computed in Example 4.11. Using the formula for the future value of an annuity, if we borrow \$2000 per year for 15 years at 8%, at the end our debt will have grown to

$$\$2000 \times \frac{1}{0.08}(1.08^{15} - 1) = \$54{,}304.23$$

After paying off our debt, we will have \$63,433.38 − \$54,304.23 = \$9139.15 remaining after 15 years.

You can also use a handheld financial calculator to do the same calculations. The calculators work in much the same way as the annuity spreadsheet. You enter any four of the five variables, and the calculator calculates the fifth variable.

CONCEPT CHECK

1. What tools can you use to simplify the calculation of present values?
2. What is the process for using the annuity spreadsheet?
3. Why do you enter some cash flows as negative and some as positive when using the spreadsheet's or financial calculator's functions?

4.7 NON-ANNUAL TIME INTERVALS

Until now we have only considered annual time intervals for our time value calculations. Do the same tools apply if we use another time interval, say a month or a day? The answer is yes; everything we have learned about time value calculations with annual time intervals applies to other time intervals so long as the following hold.

1. The interest rate used corresponds to the specific time interval.
2. The number of periods used corresponds to the specific time interval.

In general, any time interval with corresponding interest rate and number of periods can be used for a time value calculation for one single cash flow. For example, suppose you have a loan that charges 5% interest every six months (i.e., semiannually). If you have a \$1000 balance on the card today, and make no payments for one year, your future balance in one year's time will be

$$FV_n = C_0 \times (1 + r)^n = \$1000 \times (1.05)^2 = \$1102.50$$

We apply the future value formula exactly as before, but with r equal to the interest rate per six months and n equal to the number of six-month time periods. Later in the text we will discuss how to convert interest rates into equivalent rates over different time intervals. For a rate of 5% per six months with compounding every six months, the equivalent one-year interest rate with annual compounding is 10.25%. Redoing the above calculation with this equivalent rate, we get the balance in one year's time to be

$$FV_n = C_0 \times (1 + r)^n = \$1000 \times (1.1025)^1 = \$1102.50$$

So it does not matter whether we use the six-month rate or the equivalent one-year rate to do the calculation as long as we are careful to use the corresponding number of time intervals. Both calculations result in the same future value of \$1102.50.

The situation is different for time value calculations involving annuities or perpetuities. In these cases, it is *necessary* that both the interest rate and number of periods correspond to the time period between cash flows. For example, if we want to calculate the present value of an annuity of cash flows that occur every six months and last for four years, then we must use the six-month rate and the number of six-month periods that occur in four years (i.e., 8 six-month periods). Suppose the rate is 5% per six months and the semiannual cash flows are $10,000 each; then the present value can only be calculated as follows:

$$PV_0 = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) = \$10{,}000 \times \frac{1}{0.05}\left(1 - \frac{1}{(1.05)^8}\right) = \$64{,}632.13$$

Alternatively, we may use the annuity spreadsheet to solve the problem. To compute the solution, we enter the four variables we know $(NPER = 8,\ RATE = 5\%,\ PMT = -10000,\ FV = 0)$ and solve for the one we want to determine (PV) using the Excel function PV(RATE, NPER, PMT, FV). The spreadsheet here calculates a future value of $64,632.13.

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	8	5.00%		−10,000	0	
Solve for PV			**64,632.13**			= PV(0.05, 8, −100000)

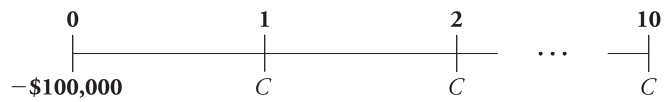
CONCEPT CHECK

1. For a single cash flow, do the present and future value formulas depend upon using a time interval of one year?
2. For time value calculations with a series of cash flows that has a non-annual time interval, what interest rate must you use? What number of periods must you use?

4.8 SOLVING FOR THE CASH FLOWS

So far, we have calculated the present value or future value of a stream of cash flows. Sometimes, however, we know the present value or future value but do not know the cash flows. The best example is a loan—you know how much you want to borrow (the present value) and you know the interest rate, but you do not know how much you need to repay each year. Suppose you are opening a business that requires an initial investment of $100,000. Your bank manager has agreed to lend you this money. The terms of the loan state that you will make equal annual payments for the next 10 years and will pay an interest rate of 8% with the first payment due one year from today. What is your annual payment?

From the bank's perspective, the timeline looks like this:



The bank will give you $100,000 today in exchange for 10 equal payments over the next decade. You need to determine the size of the payment C that the bank will require. For the bank to be willing to lend you $100,000, the loan cash flows must have a present value of $100,000 when evaluated at the bank's interest rate of 8%. That is,

$$\$100{,}000 = PV\left(\text{10-year annuity of } C \text{ per year, evaluated at the loan rate}\right)$$

Using the formula for the present value of an annuity,

$$\$100{,}000 = C \times \frac{1}{0.08}\left(1 - \frac{1}{1.08^{10}}\right) = C \times 6.71008$$

solving this equation for C gives

$$C = \frac{\$100{,}000}{6.71008} = \$14{,}902.95$$

You will be required to make 10 annual payments of \$14,902.95 in exchange for \$100,000 today.

We can also solve this problem with the annuity spreadsheet:

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	10	8.00%	100,000		0	
Solve for PMT				**−14,902.95**		= PMT(0.08, 10, 100000, 0)

In general, when solving for a loan payment, think of the amount borrowed (the loan principal) as the present value of the payments. If the payments of the loan are an annuity, we can solve for the payment of the loan by inverting the annuity formula. Writing the equation for the payments formally for a loan with principal PV, requiring n periodic payments of C and interest rate r, we have

$$PV\,(\text{annuity of } C \text{ for } n \text{ periods}) = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

Solving this equation for C gives the general formula for the loan payment in terms of the outstanding principal (amount borrowed), PV; interest rate, r; and number of payments, n:

Loan Payment

$$C = \frac{PV}{\frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right)} \tag{4.13}$$

EXAMPLE 4.14 COMPUTING A LOAN PAYMENT

Problem

Your firm plans to buy a warehouse for \$100,000. The bank offers you a 30-year loan with equal annual payments and an interest rate of 8% per year. The bank requires that your firm pay 20% of the purchase price as a down payment, so you can borrow only \$80,000. What is the annual loan payment?

Solution

We start with the timeline (from the bank's perspective):

Using Eq. 4.12, we can solve for the loan payment, C, as follows:

$$C = \frac{PV}{\frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right)} = \frac{\$80{,}000}{\frac{1}{0.08}\left(1 - \frac{1}{(1.08)^{30}}\right)}$$
$$= \$7106.19$$

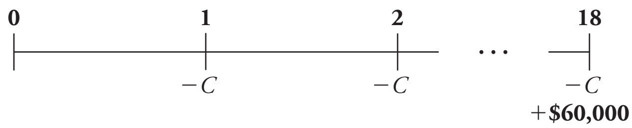
Using the annuity spreadsheet:

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	30	8.00%	−80,000		0	
Solve for PMT				**7106.19**		= PMT(0.08, 30, −80000, 0)

Your firm will need to pay $7106.19 each year to repay the loan.

We can use this same idea to solve for the cash flows when we know the future value rather than the present value. As an example, suppose you have just had a child. You decide to be prudent and start saving this year for her university education. You would like to have $60,000 saved by the time your daughter is 18 years old. If you can earn 7% per year on your savings, how much do you need to save each year to meet your goal?

The timeline for this example is



That is, you plan to save some amount C per year, and then withdraw $60,000 from the bank in 18 years. Therefore, we need to find the annuity payment that has a future value of $60,000 in 18 years. Using the formula for the future value of an annuity from Eq. 4.8,

$$\$60{,}000 = FV(\text{annuity}) = C \times \frac{1}{0.07}(1.07^{18} - 1) = C \times 33.99903$$

Therefore, C = $60,000 ÷ 33.99903 = $1764.76. So you need to save $1764.76 per year. If you do, then at a 7% interest rate your savings will grow to $60,000 by the time your child is 18 years old.

Now let's solve this problem with the annuity spreadsheet:

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	18	7.00%	0		60,000	
Solve for PMT				**−1764.76**		= PMT(0.07, 18, 0, 60000)

Once again, we find that we need to save $1764.76 for 18 years to accumulate $60,000.

CONCEPT CHECK

1. How can we solve for the required annuity payment for a loan?
2. How can we determine the required amount to save each year to reach a savings goal?

4.9 THE INTERNAL RATE OF RETURN

In some situations, you know the present value and cash flows of an investment opportunity but you do not know the interest rate that equates them. This interest rate is called the **internal rate of return (*IRR*)**, defined as the interest rate that sets the net present value of the cash flows equal to zero.

For example, suppose that you have an investment opportunity that requires a \$1000 investment today and will have a \$2000 payoff in six years. On a timeline,

One way to analyze this investment is to ask the question: What interest rate, r, would you need so that the *NPV* of this investment is zero?

$$NPV = -\$1000 + \frac{\$2000}{(1 + r)^6} = 0$$

Rearranging gives

$$\$1000 \times (1 + r)^6 = \$2000$$

That is, r is the interest rate you would need to earn on your \$1000 to have a future value of \$2000 in six years. We can solve for r as follows:

$$1 + r = \left(\frac{\$2000}{\$1000}\right)^{1/6} = 1.12246205$$

or $r = 12.246205\%$. This rate is the *IRR* of this investment opportunity. Making this investment is like earning 12.246205% per year on your money for six years.

When there are just two cash flows, as in the preceding example, it is easy to compute the *IRR*. Consider the general case in which you invest an amount P today, and receive FV in N years. Then the *IRR* satisfies the equation $P \times (1 + IRR)^n = FV$, which implies

$$IRR \text{ with two cash flows} = (FV/P)^{1/n} - 1 \tag{4.14}$$

Note in the formula that we take the total return of the investment over n years, FV/P, and convert it to an equivalent one-year return by raising it to the power $1/n$.

The *IRR* is also straightforward to calculate for a perpetuity, as we demonstrate in the next example.

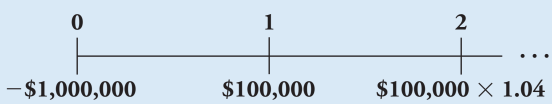
EXAMPLE 4.15 COMPUTING THE *IRR* FOR A PERPETUITY

Problem

Jessica has just graduated with her MBA. Rather than take the job she was offered at Scotia Capital she has decided to go into business for herself. She believes that her business will require an initial investment of \$1 million. After that it will generate a cash flow of \$100,000 at the end of one year, and this amount will grow by 4% per year thereafter. What is the *IRR* of this investment opportunity?

Solution

The timeline is



The timeline shows that the future cash flows are a growing perpetuity with a growth rate of 4%. Recall from Eq. 4.10 that the PV of a growing perpetuity is $C_1/(r - g)$. Thus, the *NPV* of this investment would equal zero if

$$\$1{,}000{,}000 = \frac{\$100{,}000}{r - 0.04}$$

We can solve this equation for r

$$r = \frac{\$100{,}000}{\$1{,}000{,}000} + 0.04 = 0.14$$

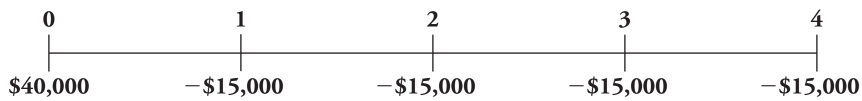
So, the *IRR* on this investment is 14%.

More generally, if we invest P and receive a perpetuity with initial cash flow C_1 and growth rate g, we can use the growing perpetuity formula to determine

$$IRR \text{ of growing perpetuity} = (C_1/P) + g \tag{4.15}$$

Now let's consider a more sophisticated example. Suppose your firm needs to purchase a new forklift. The dealer gives you two options: (1) a price for the forklift if you pay cash and (2) the annual payments if you take out a loan from the dealer. To evaluate the loan that the dealer is offering you, you will want to compare the rate on the loan with the rate that your bank is willing to offer you. Given the loan payment that the dealer quotes, how do you compute the interest rate charged by the dealer?

In this case, we need to compute the *IRR* of the dealer's loan. Suppose the cash price of the forklift is $40,000, and the dealer offers financing with no down payment and four annual payments of $15,000. This loan has the following timeline:



From the timeline it is clear that the loan is a four-year annuity with a payment of $15,000 per year and a present value of $40,000. Setting the *NPV* of the cash flows equal to zero requires that the present value of the payments equals the purchase price:

$$40{,}000 = 15{,}000 \times \frac{1}{r}\left(1 - \frac{1}{(1 + r)^4}\right)$$

The value of r that solves this equation, the *IRR*, is the interest rate charged on the loan. Unfortunately, in this case, there is no simple way to solve for the interest rate r.[8] The only way to solve this equation is to guess values of r until you find the right one.

Start by guessing $r = 10\%$. In this case, the value of the annuity is

$$\$15{,}000 \times \frac{1}{0.10}\left(1 - \frac{1}{(1.10)^4}\right) = \$47{,}548$$

8. With five or more periods and general cash flows, there is *no* general formula to solve for r; trial and error (by hand or computer) is the *only* way to compute the *IRR*.

The present value of the payments is too large. To lower it, we need to use a higher interest rate. We guess 20% this time:

$$\$15{,}000 \times \frac{1}{0.20}\left(1 - \frac{1}{(1.20)^4}\right) = \$38{,}831$$

Now the present value of the payments is too low, so we must pick a rate between 10% and 20%. We continue to guess until we find the right rate. Let us try 18.45%:

$$\$15{,}000 \times \frac{1}{0.1845}\left(1 - \frac{1}{(1.1845)^4}\right) = \$40{,}000$$

The interest rate charged by the dealer is about 18.45%.

An easier solution than guessing the *IRR* and manually calculating values is to use a spreadsheet or calculator to automate the guessing process. When the cash flows are an annuity, as in this example, we can use the annuity spreadsheet in Excel to compute the *IRR*. Recall that the annuity spreadsheet solves Eq. 4.12. It ensures that the *NPV* of investing in the annuity is zero. When the unknown variable is the interest rate, it will solve for the interest rate that sets the *NPV* equal to zero—that is, the *IRR*. For this case,

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	4		40,000	−15,000	0	
Solve for Rate		**18.45%**				= RATE(4, −15,000, 40000, 0)

The annuity spreadsheet correctly computes an *IRR* of 18.45% or, if we show more decimal places, 18.450489%—an amount we are unlikely to guess very quickly!

EXAMPLE 4.16 COMPUTING THE INTERNAL RATE OF RETURN FOR AN ANNUITY

Problem

Scotia Capital was so impressed with Jessica that it has decided to fund her business. In return for providing the initial capital of $1 million, Jessica has agreed to pay them $125,000 at the end of each year for the next 30 years. What is the internal rate of return on Scotia Capital's investment in Jessica's company, assuming she fulfills her commitment?

Solution

Here is the timeline (from Scotia Capital's perspective):

0	1	2	. . .	30
−$1,000,000	**$125,000**	**$125,000**		**$125,000**

The timeline shows that the future cash flows are a 30-year annuity. Setting the *NPV* equal to zero requires

$$\$1{,}000{,}000 = \$125{,}000 \times \frac{1}{r}\left(1 - \frac{1}{(1 + r)^{30}}\right)$$

Using the annuity spreadsheet to solve for r,

	NPER	RATE	PV	PMT	FV	Excel Formula
Given	30		−1,000,000	125,000	0	
Solve for Rate		**12.09%**				= RATE(30, 125000, −1000000,0)

The *IRR* on this investment is 12.093041%. In this case, we can interpret the *IRR* of 12.093041% as the effective interest rate of the loan.

CONCEPT CHECK

1. What is the internal rate of return (*IRR*)?
2. In what two cases is the internal rate of return easy to calculate?

EXCEL'S *IRR* FUNCTION

Excel also has a built in function, *IRR*, that will calculate the *IRR* of a stream of cash flows. Excel's *IRR* function has the format, *IRR*(values, guess), where "values" is the range containing the cash flows, and "guess" is an optional starting guess where Excel begins its search for an *IRR*. See the example below:

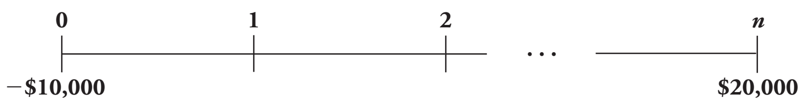
	A	B	C	D	E
1	Period	0	1	2	3
2	Cash Flow C_t	(1,000.0)	300.0	400.0	500.0
3	IRR	8.9% =IRR(B2:E2)			

There are three things to note about the *IRR* function. First, the values given to the *IRR* function should include all of the cash flows of the project, including the one at date 0. In this sense, the *IRR* and *NPV* functions in Excel are inconsistent. Second, like the *NPV* function, the *IRR* ignores the period associated with any blank cells. Finally, as we will learn later, in some settings the *IRR* function may fail to find a solution, or may give a different answer depending on the initial guess.

4.10 SOLVING FOR THE NUMBER OF PERIODS

In addition to solving for cash flows or the interest rate, we can solve for the amount of time it will take a sum of money to grow to a known value. In this case, the interest rate, present value, and future value are all known. We need to compute how long it will take for the present value to grow to the future value.

Suppose we invest $10,000 in an account paying 10% interest, and we want to know how long it will take for the amount to grow to $20,000.



We want to determine n.

In terms of our formulas, we need to find n so that the future value of our investment equals $20,000:

$$FV = \$10{,}000 \times 1.10^n = \$20{,}000 \tag{4.16}$$

This problem can be solved on the annuity spreadsheet. In this case, we solve for n:

	NPER	RATE	PV	PMT	FV	Excel Formula
Given		10.00%	−10,000	0	20,000	
Solve for NPER	**7.27254**					= NPER(0.10, 0, −10000, 20000)

It will take about 7.27 years for our savings to grow to \$20,000.

This problem can also be solved mathematically. Dividing both sides of Eq. 4.16 by \$10,000, we have

$$1.10^n = 20{,}000 \div 10{,}000 = 2$$

To solve for an exponent, we take the logarithm of both sides, and use the fact that

$$\ln(x^y) = y \times \ln(x)$$

$$n \times \ln(1.10) = \ln(2)$$

$$n = \ln(2)/\ln(1.10) = 0.693147/0.095310 = 7.27254 \text{ years}$$

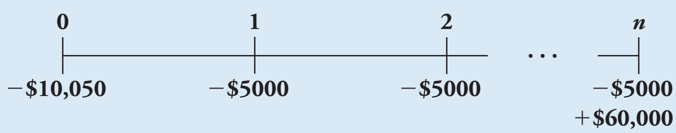
EXAMPLE 4.17 SOLVING FOR THE NUMBER OF PERIODS IN A SAVINGS PLAN

Problem

You are saving to make a down payment on a house. You have \$10,050 saved already, and you can afford to save an additional \$5000 per year at the end of each year. If you earn 7.25% per year on your savings, how long will it take you to save \$60,000?

Solution

The timeline for this problem is



We need to find n so that the future value of our current savings plus the future value of our planned additional savings (which is an annuity) equals our desired amount:

$$\$10{,}050 \times 1.0725^n + \$5000 \times \frac{1}{0.0725}(1.0725^n - 1) = \$60{,}000$$

To solve mathematically, rearrange the equation to

$$1.0725^n = \frac{\$60{,}000 \times 0.0725 + \$5000}{\$10{,}050 \times 0.0725 + \$5000} = 1.632$$

We can then solve for n:

$$n = \frac{\ln(1.632)}{\ln(1.0725)} \approx 7 \text{ years}$$

It will take about seven years to save the down payment. We can also solve this problem using the annuity spreadsheet:

	NPER	RATE	PV	PMT	FV	Excel Formula
Given		7.25%	−10,050	−5000	60,000	= NPER(0.0725,
Solve for N	**6.999346**					−5000, −10050, 60000)

Because we avoided some rounding in the spreadsheet, we see the result calculated as 6.999346 years. Thus, if we have seven years for our current savings plan, then we will have slightly more than $60,000 saved due to the slightly longer time period to earn interest.

In this chapter, we developed the tools a financial manager needs to apply the *NPV* rule when cash flows occur at different points in time. As we have seen, the interest rate we use to discount or compound the cash flows is a critical input to any of our present or future value calculations. Throughout the chapter, we have taken the interest rate as given.

What determines the interest rate that we should use when discounting cash flows? The Law of One Price implies that we must rely on market information to assess the value of cash flows across time. In the next chapter, we learn the drivers of market interest rates as well as how they are quoted. Understanding interest rate quoting conventions will also allow us to extend the tools we developed in this chapter to situations where the cash flows are paid, and interest is compounded, more or less than once per year.

CONCEPT CHECK

1. How do you solve for the cash flow of an annuity?
2. What is the internal rate of return, and how do you calculate it?
3. How do you solve for the number of periods to pay off an annuity?

SUMMARY

1. Timelines are a critical first step in organizing the cash flows in a financial problem.
2. There are three rules of time travel:
   a. Only cash flow values that occur at the same point in time can be compared or combined.
   b. To move a cash flow forward in time, you must compound it.
   c. To move a cash flow backward in time, you must discount it.
3. The future value in n years of a cash flow C today is

$$FV_n = C_0 \times (1 + r)^n \tag{4.1}$$

4. The present value today of a cash flow C received in n years is

$$PV_0 = \frac{C_n}{(1 + r)^n} \tag{4.2}$$

5. The present value of a cash flow stream is

$$PV_0 = \sum_{t=0}^{n} \frac{C_t}{(1 + r)^t} \tag{4.3}$$

6. The future value on date n of a cash flow stream with a present value of PV_0 is

$$FV_n = PV_0 \times (1 + r)^n \tag{4.4}$$

7. The *NPV* of an investment opportunity is *PV* (benefits − costs).

8. A regular perpetuity is a constant cash flow C that starts in one period and is paid every period, forever. The present value of a perpetuity is

$$PV_0 = \frac{C}{r} \tag{4.5}$$

9. An regular annuity is a constant cash flow C that starts in one period and is paid every period for n periods. The present value of an annuity is

$$PV_0 = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) \tag{4.7}$$

The future value of an annuity at the end of the annuity is

$$FV_n = C \times \frac{1}{r}((1+r)^n - 1) \tag{4.8}$$

10. In a growing perpetuity or annuity, the cash flows start in one period and grow at a constant rate g each period. The present value of a growing perpetuity is

$$PV_0 = \frac{C_1}{r-g} \tag{4.9}$$

The present value of a growing annuity is

$$PV_0 = \frac{C_1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right] \tag{4.10}$$

The future value of a growing annuity at the end of the growing annuity is

$$FV_n = \frac{C_1}{r-g}[(1+r)^n - (1+g)^n] \tag{4.11}$$

11. The annuity and perpetuity formulas can be used to solve for the annuity payments when either the present value or the future value is known. The periodic payment on an n-period loan with principal PV and interest rate r is

$$C = \frac{PV}{\frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right)} \tag{4.13}$$

12. The internal rate of return (IRR) of an investment opportunity is the interest rate that sets the NPV of the investment opportunity equal to zero.

13. The annuity formulas can be used to solve for the number of periods it takes to save a fixed amount of money.

KEY TERMS

PROBLEMS

MyFinanceLab **Visit MyFinanceLab for the problems indicated in red below.**

All problems are available in MyFinanceLab. An asterisk (*) indicates problems with a higher level of difficulty.

The Timeline

1. You have just taken out a five-year loan from a bank to buy an engagement ring. The ring costs \$5000. You plan to put down \$1000 and borrow \$4000. You will need to make annual payments of \$1000 at the end of each year. Show the timeline of the loan from your perspective. How would the timeline differ if you created it from the bank's perspective?

2. You currently have a four-year-old mortgage outstanding on your house. You make monthly payments of \$1500. You have just made a payment. The mortgage has 26 years to go (i.e., it had an original term of 30 years). Show the timeline from your perspective. How would the timeline differ if you created it from the bank's perspective?

The Three Rules of Time Travel

3. Calculate the future value of \$2000 in
   a. five years at an interest rate of 5% per year.
   b. 10 years at an interest rate of 5% per year.
   c. five years at an interest rate of 10% per year.
   d. Why is the amount of interest earned in part a less than half the amount of interest earned in part b?

4. What is the present value of \$10,000 received
   a. 12 years from today when the interest rate is 4% per year?
   b. 20 years from today when the interest rate is 8% per year?
   c. six years from today when the interest rate is 2% per year?

5. Your brother has offered to give you either \$5000 today or \$10,000 in 10 years. If the interest rate is 7% per year, which option is preferable?

6. Consider the following alternatives:
      i. \$100 received in one year
      ii. \$200 received in five years
      iii. \$300 received in 10 years
   a. Rank the alternatives from most valuable to least valuable if the interest rate is 10% per year.
   b. What is your ranking if the interest rate is only 5% per year?
   c. What is your ranking if the interest rate is 20% per year?

7. Suppose you invest \$1000 in an account paying 8% interest per year.
   a. What is the balance in the account after three years? How much of this balance corresponds to "interest on interest"?
   b. What is the balance in the account after 25 years? How much of this balance corresponds to interest on interest?

8. Your daughter is currently eight years old. You anticipate that she will be going to university in 10 years. You would like to have \$100,000 in a registered education savings plan (RESP) to fund her education at that time. If the account promises to pay a fixed interest rate of 3%

per year, how much money do you need to put into the account today (ignoring government grants) to ensure that you will have $100,000 in 10 years?

9. You are thinking of retiring. Your retirement plan will pay you either $250,000 immediately on retirement or $350,000 five years after the date of your retirement. Which alternative should you choose if the interest rate is

a. 0% per year?

b. 8% per year?

c. 20% per year?

10. Your grandfather put some money in an account for you on the day you were born. You are now 18 years old and are allowed to withdraw the money for the first time. The account currently has $3996 in it and pays an 8% interest rate.

a. How much money would be in the account if you left the money there until your 25th birthday?

b. What if you left the money until your 65th birthday?

c. How much money did your grandfather originally put in the account?

Valuing a Stream of Cash Flows

11. Suppose you receive $100 at the end of each year for the next three years.

a. If the interest rate is 8%, what is the present value of these cash flows?

b. What is the future value in three years of the present value you computed in part a?

c. Suppose you deposit the cash flows in a bank account that pays 8% interest per year. What is the balance in the account at the end of each of the next three years (after your deposit is made)? How does the final bank balance compare with your answer in part b?

EXCEL

12. You have just received a windfall from an investment you made in a friend's business. He will be paying you $10,000 at the end of this year, $20,000 at the end of the following year, and $30,000 at the end of the year after that (three years from today). The interest rate is 3.5% per year.

a. What is the present value of your windfall?

b. What is the future value of your windfall in three years (on the date of the last payment)?

EXCEL

13. You have a loan outstanding. It requires making three annual payments at the end of the next three years of $1000 each. Your bank has offered to allow you to skip making the next two payments in lieu of making one large payment at the end of the loan's term in three years. If the interest rate on the loan is 5%, what final payment will the bank require you to make so that it is indifferent between the two forms of payment?

Calculating the Net Present Value

EXCEL

14. You have been offered a unique investment opportunity. If you invest $10,000 today, you will receive $500 one year from now, $1500 two years from now, and $10,000 ten years from now.

a. What is the *NPV* of the opportunity if the interest rate is 6% per year? Should you take the opportunity?

b. What is the *NPV* of the opportunity if the interest rate is 2% per year? Should you take it now?

EXCEL

15. Magda Nowak owns her own business and is considering an investment. If she undertakes the investment, it will pay $4000 at the end of each of the next three years. The opportunity requires an initial investment of $1000 plus an additional investment at the end of the second year of $5000. What is the *NPV* of this opportunity if the interest rate is 2% per year? Should Magda take it?

Perpetuities and Annuities

16. Your buddy in mechanical engineering has invented a money machine. The main drawback of the machine is that it is slow. It takes one year to manufacture $100. However, once built, the machine will last forever and will require no maintenance. The machine can be built immediately, but it will cost $1000 to build. Your buddy wants to know if he should invest the money to construct it. If the interest rate is 9.5% per year, what should your buddy do?

17. How would your answer to Problem 16 change if the machine takes one year to build?

18. The British government has a consol bond outstanding paying £100 per year forever. Assume the current interest rate is 4% per year.

a. What is the value of the bond immediately after a payment is made?

b. What is the value of the bond immediately before a payment is made?

19. What is the present value of $1000 paid at the end of each of the next 100 years if the interest rate is 7% per year?

***20.** You are head of the Schwartz Family Endowment for the Arts. You have decided to fund an arts school in Toronto in perpetuity. Every five years, you will give the school $1 million. The first payment will occur five years from today. If the interest rate is 8% per year, what is the present value of your gift?

21. You are the beneficiary of a trust fund that will start paying you cash flows in five years. The cash flows will be $25,000 per year and will continue for 40 years. If the interest rate is 4% per year, what is the value needed in the trust fund now to fund these cash flows?

22. You are 25 years old and decide to start saving for your retirement. You plan to save $5000 at the end of each year (so the first deposit will be one year from now), and will make the last deposit when you retire at age 65. Suppose you earn 8% per year on your retirement savings.

a. How much will you have saved for retirement?

b. How much will you have saved if you wait until age 35 to start saving (again, with your first deposit at the end of the year)?

EXCEL **23.** Your grandmother has been putting $1000 into a savings account on every birthday since your first (that is, when you turned one). The account pays an interest rate of 3%. How much money will be in the account on your 18th birthday immediately after your grandmother makes the deposit on that birthday?

EXCEL **24.** A rich relative has bequeathed you a growing perpetuity. The first payment will occur in a year and will be $1000. Each year after that, you will receive a payment on the anniversary of the last payment that is 8% larger than the last payment. This pattern of payments will go on forever. If the interest rate is 12% per year,

a. What is today's value of the bequest?

b. What is the value of the bequest immediately after the first payment is made?

***25.** You are thinking of building a new machine that will save you $1000 in the first year. The machine will then begin to wear out so that the savings *decline* at a rate of 2% per year forever. What is the present value of the savings if the interest rate is 5% per year?

26. You work for a pharmaceutical company that has developed a new drug. The patent on the drug will last 17 years. You expect that the drug's profits will be $2 million in its first year and that this amount will grow at a rate of 5% per year for the next 17 years. Once the patent expires, other pharmaceutical companies will be able to produce the same drug and competition will likely drive profits to zero. What is the present value of the new drug if the interest rate is 10% per year?

EXCEL **27.** Your oldest daughter is about to start kindergarten at a private school. Tuition is \$10,000 per year, payable at the *beginning* of the school year. You expect to keep your daughter in private school through high school. You expect tuition to increase at a rate of 5% per year over the 13 years of her schooling. What is the present value of the tuition payments if the interest rate is 5% per year? How much would you need to have in the bank now to fund all 13 years of tuition?

EXCEL **28.** A rich aunt has promised you \$5000 one year from today. In addition, each year after that, she has promised you a payment (on the anniversary of the last payment) that is 5% larger than the last payment. She will continue to show this generosity for 20 years, giving a total of 20 payments. If the interest rate is 5%, what is her promise worth today?

EXCEL ***29.** You are running a hot Internet pharmacy company. Analysts predict that its earnings will grow at 30% per year for the next five years. After that, as competition increases, earnings growth is expected to slow to 2% per year and continue at that level forever. Your company has just announced earnings of \$1,000,000. What is the present value of all future earnings if the interest rate is 8%? (Assume all cash flows occur at the end of the year.)

Solving Problems with a Spreadsheet

EXCEL ***30.** Your brother has offered to give you \$100, starting next year, and after that growing at 3% for the next 20 years. You would like to calculate the value of this offer by calculating how much money you would need to deposit in the local bank so that the account will generate the same cash flows as he is offering you. Your local bank will guarantee a 6% annual interest rate so long as you have money in the account.

a. How much money will you need to deposit into the account today?

b. Using an Excel spreadsheet, show explicitly that you can deposit this amount of money into the account, and every year withdraw what your brother has promised, leaving the account with nothing after the last withdrawal.

Non-Annual Time Intervals

31. You have just put \$100 in the bank and your account earns 1% interest every month with monthly compounding.

a. How much will be in your account after one year (show 6 decimal places)?

b. If your bank changed to paying interest only once per year, what yearly rate would give you the same amount as what you calculated in part a?

***32.** Suppose you set up a savings plan whereby you will deposit \$1000 per month into an account earning 0.5% per month compounded monthly. Your first deposit will be one month from now and your last deposit will be five years from now. How much will be in your account immediately after your last deposit?

Solving for the Cash Flows

33. You have decided to buy a perpetuity. The bond makes one payment at the end of every year forever and has an interest rate of 5%. If you initially put \$1000 into the bond, what is the payment every year?

34. You are purchasing a house and your bank is giving you a special mortgage that will require annual payments for 25 years. The amount borrowed now is \$300,000 and the first mortgage payment will be in one year.

a. Using C as the payment amount, indicate on a timeline all of the *cash flows* from your perspective related to this mortgage (outflows should be indicated as negative numbers).

b. What will your payments be if the interest rate is 3% per year?

c. What will your payments be if the interest rate is 4% per year?

d. Comparing your answers in parts b and c, when the interest rate increased by 1%, by what percent did the mortgage payment increase?

***35.** You are thinking about buying a piece of art that costs $50,000. The art dealer is proposing the following deal: He will lend you the money, and you will repay the loan by making the same payment every two years for the next 20 years (i.e., a total of 10 payments). If the interest rate is 4%, how much will you have to pay every two years?

EXCEL **36.** You are saving for retirement. To live comfortably, you decide you will need to save $2 million in your RRSP by the time you are 65. Today is your 30th birthday, and you decide that, starting today and continuing on every birthday up to and including your 65th birthday, you will put the same amount into an RRSP account. If the interest rate is 5%, how much must you set aside each year to make sure that you will have $2 million in the RRSP on your 65th birthday?

EXCEL ***37.** You realize that the plan in Problem 36 has a flaw. Because your income will increase over your lifetime, it would be more realistic to save less now and more later. Instead of putting the same amount aside each year, you decide to let the amount that you set aside grow by 7% per year. Under this plan, how much will you put into the account today? (Recall that you are planning to make the first contribution to the RRSP account today.)

***38.** You are 35 years old, and decide to save $5000 each year (with the first deposit one year from now), in an account paying 8% interest per year. You will make your last deposit 30 years from now when you retire at age 65. During retirement, you plan to withdraw funds from the account at the end of each year (so your first withdrawal is at age 66). What constant amount will you be able to withdraw each year if you want the funds to last until you are 90?

EXCEL ***39.** You have just turned 30 years old, have just received your MBA, and have accepted your first job. Now you must decide how much money to put into your RRSP. Your RRSP works as follows: Every dollar in the plan earns 7% per year. You cannot make withdrawals until your 65th birthday. After that point, you can make withdrawals as you see fit. You decide that you will plan to live to 100 and work until you turn 65. You estimate that to live comfortably in retirement, you will need $100,000 per year starting at the end of the first year of retirement (i.e., when you turn 66) and ending on your 100th birthday. You will contribute the same amount to the plan at the end of every year that you work. How much do you need to contribute each year to fund your retirement?

EXCEL ***40.** Problem 39 is not very realistic because most people do not contribute a fixed amount to their RRSP each year. Instead, you would prefer to contribute a fixed percentage of your salary each year. Assume that your starting salary is $75,000 per year and it will grow 2% per year until you retire. Assuming everything else stays the same as in Problem 39, what percentage of your income do you need to contribute to the plan every year to fund the same retirement income?

The Internal Rate of Return

41. Suppose you invest $2000 today and receive $10,000 in five years.

a. What is the *IRR* of this opportunity?

b. Suppose another investment opportunity also requires $2000 upfront, but pays an equal amount at the end of each year for the next five years. If this investment has the same *IRR* as the first one, what is the amount you will receive each year?

42. You are shopping for a car and read the following advertisement in the newspaper: "Own a new Spitfire! No money down. Four annual payments of just $10,000." You have shopped around and know that you can buy a Spitfire for cash for $32,500. What is the interest rate the dealer is advertising (what is the *IRR* of the loan in the advertisement)? Assume that you must make the annual payments at the end of each year.

43. A local bank is running the following advertisement in the newspaper: "For just $1000 we will pay you $100 forever!" The fine print in the ad says that for a $1000 deposit, the bank will pay $100 every year in perpetuity, starting one year after the deposit is made. What interest rate is the bank advertising (what is the *IRR* of this investment)?

EXCEL ***44.** The Laiterie de Coaticook in the Eastern Townships of Quebec produces several types of cheddar cheese. It sells the cheese in four varieties: aged 2 months, 9 months, 15 months, and 2 years. At the producer's store, it sells 2 pounds of each variety for the following prices: $7.95, $9.49, $10.95, and $11.95, respectively. Consider the cheese maker's decision whether to continue to age a particular 2-pound block of cheese. At 2 months, he can either sell the cheese immediately or let it age further. If he sells it now, he will receive $7.95 immediately. If he ages the cheese, he must give up the $7.95 today to receive a higher amount in the future. What is the *IRR* (expressed in percent per month) of the investment of giving up $79.50 today by choosing to store 20 pounds of cheese that is currently two months old and instead selling 10 pounds of this cheese when it has aged nine months, 6 pounds when it has aged 15 months, and the remaining 4 pounds when it has aged two years?

Solving for the Number of Periods

***45.** Your grandmother bought an annuity from Manulife Financial for $200,000 when she retired. In exchange for the $200,000, Manulife will pay her $25,000 per year until she dies. The interest rate is 5%. How long must she live after the day she retired to come out ahead (that is, to get more in *value* than what she paid in)?

EXCEL ***46.** You are thinking of making an investment in a new plant. The plant will generate revenues of $1 million per year for as long as you maintain it. You expect that the maintenance cost will start at $50,000 per year and will increase 5% per year thereafter. Assume that all revenue and maintenance costs occur at the end of the year. You intend to run the plant as long as it continues to make a positive cash flow (as long as the cash generated by the plant exceeds the maintenance costs). The plant can be built and become operational immediately. If the plant costs $10 million to build, and the interest rate is 6% per year, should you invest in the plant?

CHAPTER 4
APPENDIX

USING A FINANCIAL CALCULATOR

Specifying Decimal Places

Make sure you always have plenty of decimal places displayed!

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Toggling Between the Beginning and End of a Period

You should always make sure that your calculator is in *end-of-period* mode.

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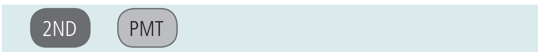
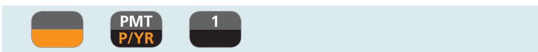


Set the Number of Periods per Year

You will avoid a lot of confusion later if you always set your periods per year "P/Y" to 1:

HP-10BII



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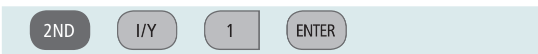

General TVM Buttons

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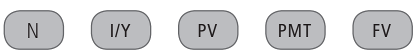

Solving for the Present Value of a Single Future Cash Flow (Example 4.1)

You are considering investing in a Government of Canada bond that will make one payment of $15,000 in 10 years. If the competitive market interest rate is fixed at 6% per year, what is the bond worth today? [Answer: $8375.92]

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[Orange Shift] C / C ALL	Press [Orange Shift] and then the [C] button to clear all previous entries.
1 0 N	Enter the Number of periods.
6 I/YR	Enter the market annual interest rate.
1 5 0 0 0 FV	Enter the Value you will recieve in 10 periods.
0 PMT	Indicate that there are no payments.
PV	Solve for the Present Value.

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2ND FV	Press [2ND] and then the [FV] button to clear all previous entries.
3 0 N	Enter the Number of periods.
1 0 I/Y	Enter the market annual interest rate.
1 0 0 0 0 PMT	Enter the payment amount per period.
0 PV	Indicate that there is no initial amount in the retirement account.
CPT FV	Solve for the Future Value.

Solving for the Future Value of an Annuity (Example 4.7)

Ellen is 35 years old, and she has decided it is time to plan seriously for her retirement. At the end of each year until she is 65, she will save $10,000 in a retirement account. If the account earns 10% per year, how much will Ellen have saved at age 65? [Answer: $1,644,940]

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[Orange Shift] C / C ALL	Press [Orange Shift] and then the [C] button to clear all previous entries.
3 0 N	Enter the Number of periods.
1 0 I/YR	Enter the market annual interest rate.
1 0 0 0 0 PMT	Enter the Payment amount per period.
0 PV	Indicate that there is no initial amount in the retirement account.
FV	Solve for the Future Value.

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2ND FV	Press [2ND] and then the [FV] button to clear all previous entries.
1 0 N	Enter the Number of periods.
6 I/Y	Enter the market annual interest rate.
1 5 0 0 0 FV	Enter the Value you will recieve in 10 periods.
0 PMT	Indicate that there are no payments.
CPT PV	Solve for the Present Value.

Solving for the Internal Rate of Return

If you have an initial cash outflow of $2000 and one cash inflow per year for the following four years of $1000, $400, $400, and $800, what is the internal rate of return on the project per year? [Answer: 12.12%]

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C / C ALL	Press [Orange Shift] and then the [C] button to clear all previous entries.
2 0 0 0 +/− CFj	Enter the initial cash outflow.
1 0 0 0 CFj	Enter the first cash inflow.
4 0 0 CFj	Enter the second cash inflow.
2 CFj	Enter the number of consecutive periods the second cash inflow occurs.
8 0 0 CFj / Nj	Enter the fourth cash inflow.
CST / IRR/YR	Press [Orange Shift] and then the [CST] button to calculate the IRR/year.

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CF	Access Cash Flow Worksheet.
2ND CE/C	Press [2ND] and then the [CE/C] button to clear all previous entries.
2 0 0 0 +/− ENTER	Enter the initial cash outflow.
↓ 1 0 0 0 ENTER	Enter the first cash inflow.
↓	Leave the frequency of the initial cash inflow at 1 (Default Setting).
↓ 4 0 0 ENTER	Enter the second cash inflow.
↓ 2 ENTER	Enter the frequency of the second cash inflow as 2.
↓ 8 0 0 ENTER	Enter the fourth cash inflow.
↓	Leave the frequency of the fourth cash inflow at 1 (Default Setting).
IRR CPT	Solve for the IRR.